THE GROTHENDIECK CONSTANT IS STRICTLY SMALLER THAN KRIVINE'S BOUND

MARK BRAVERMAN, KONSTANTIN MAKARYCHEV, YURY MAKARYCHEV, AND ASSAF NAOR

Abstract. The (real) Grothendieck constant K_G is the infimum over those $K \in (0,\infty)$ such that for every $m,n \in \mathbb{N}$ and every $m \times n$ real matrix (a_{ij}) we have

$$\max_{\{x_i\}_{i=1}^m,\{y_j\}_{j=1}^n \subseteq S^{n+m-1}} \sum_{i=1}^m \sum_{j=1}^n a_{ij}\langle x_i, y_j\rangle \leqslant K \max_{\{\varepsilon_i\}_{i=1}^m,\{\delta_j\}_{j=1}^n \subseteq \{-1,1\}} \sum_{i=1}^m \sum_{j=1}^n a_{ij}\varepsilon_i\delta_j.$$

The classical Grothendieck inequality asserts the non-obvious fact that the above inequality does hold true for some $K \in (0,\infty)$ that is independent of m,n and (a_{ij}). Since Grothendieck's 1953 discovery of this powerful theorem, it has found numerous applications in a variety of areas, but despite attracting a lot of attention, the exact value of the Grothendieck constant K_G remains a mystery. The last progress on this problem was in 1977, when Krivine proved that $K_G \leqslant \frac{\pi}{2\log\left(1+\sqrt{2}\right)}$ and conjectured that his bound is optimal. Krivine's conjecture has been restated repeatedly since 1977, resulting in focusing the subsequent research on the search for examples of matrices (a_{ij}) which exhibit (asymptotically, as $m,n\to\infty$) a lower bound on K_G that matches Krivine's bound. Here we obtain an improved Grothendieck inequality that holds for *all* matrices (a_{ij}) and yields a bound $K_G < \frac{\pi}{2\log\left(1+\sqrt{2}\right)} - \varepsilon_0$ for some effective constant $\varepsilon_0 > 0$. Other than disproving Krivine's conjecture, and along the way also disproving an intermediate conjecture of König that was made in 2000 as a step towards Krivine's conjecture, our main contribution is conceptual: despite dealing with a binary rounding problem, random 2-dimensional projections combined with a careful partition of $\mathbb{R}^2$ in order to round the projected vectors to values in $\{-1,1\}$, perform better than the ubiquitous random hyperplane technique. By establishing the usefulness of higher dimensional rounding schemes, this fact has consequences in approximation algorithms. Specifically, it yields the best known polynomial time approximation algorithm for the Frieze-Kannan Cut Norm problem, a generic and well-studied optimization problem with many applications.

1. Introduction

In his 1953 Resumé [14], Grothendieck proved a theorem that he called "le théorème fondamental de la théorie metrique des produits tensoriels". This result is known today as Grothendieck's inequality. An equivalent formulation of Grothendieck's inequality, due to Lindenstrauss and Pełczyński [26], states that there exists a universal constant $K \in (0,\infty)$ such that for every $m,n \in \mathbb{N}$, every $m \times n$ matrix (a_{ij}) with real entries, and every $m+n$ unit

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vectors $x_1,\dots,x_m,y_1,\dots,y_n \in S^{m+n-1}$, there exist $\varepsilon_1,\dots,\varepsilon_m,\delta_1,\dots,\delta_n \in \{-1,1\}$ satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i,y_j\rangle \leqslant K\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varepsilon_i\delta_j. \tag{1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^{m+n}$. The infimum over those $K \in (0,\infty)$ for which (1) holds true is called the Grothendieck constant, and is denoted K_G.

Grothendieck's inequality is important to several disciplines, including the geometry of Banach spaces, C^* algebras, harmonic analysis, operator spaces, quantum mechanics, and computer science. Rather than attempting to explain the ramifications of Grothendieck's inequality, we refer to the books [27, 36, 31, 17, 10, 5, 13, 1, 9] and especially Pisier's recent survey [32]. The survey [21] is devoted to Grothendieck's inequality in computer science; Section 2 below contains a brief discussion of this topic.

Problem 3 of Grothendieck's Resumé asks for the determination of the exact value of K_G. This problem remains open despite major effort by many mathematicians. In fact, even though K_G occurs in numerous mathematical theorems, and has equivalent interpretations as a key quantity in physics [37, 11] and computer science [2, 33], we currently do not even know what is the second digit of K_G; the best known bounds [23, 34] are $K_G \in (1.676, 1.782)$.

Following the upper bounds on K_G obtained in [14, 26, 35] (see also the alternative proofs of (1) in [28, 29, 30, 6, 10, 18], yielding worse bounds on K_G), progress on Grothendieck's Problem 3 halted after a beautiful 1977 theorem of Krivine [23], who proved that

$$K_G \leqslant \frac{\pi}{2\log\left(1+\sqrt{2}\right)}\ (= 1.782...). \tag{2}$$

One reason for this lack of improvement since 1977 is that Krivine conjectured [23] that his bound is actually the exact value of K_G. Krivine's conjecture has become the leading conjecture in this area, and as such it has been restated repeatedly in subsequent publications; see for example [24, 30, 31, 22, 7]. The belief[1] that the estimate (2) is the exact value of K_G focused research on finding examples of matrices (a_{ij}) that exhibit a matching lower bound on K_G. Following work of Haagerup, Tomczak-Jaegermann and König, the search for such matrices led in 2000 to a clean intermediate conjecture of König [22], on maximizers of a certain oscillatory integral operator, that was shown to imply Krivine's conjecture; we will explain this conjecture, which we resolve in this paper, in Section 1.1 below. Here we prove that Krivine's conjecture is false, thus obtaining the best known upper bound on K_G.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that*

$$K_G < \frac{\pi}{2\log\left(1+\sqrt{2}\right)} - \varepsilon_0.$$

We stress that our proof is effective, and it readily yields a concrete positive lower bound on ε_0. We chose not to state an explicit new upper bound on the Grothendieck constant since we know that our estimate is suboptimal. Section 3 below contains a discussion of potential improvements of our bound, based on challenging open problems that conceivably might even lead to an exact evaluation of K_G.

[1]E.g., quoting Pisier's book [31, p. 64], "The best known estimate for K_G is due to Krivine, who proved (cf. [Kri3]) that $K_G \leqslant \frac{2}{\log\left(1+\sqrt{2}\right)} = 1.782...$ and conjectured that this is the exact value of K_G . . . Krivine claims that he checked $K_G > \pi/2$, and he has convincing (unpublished) evidence that his bound is sharp."

Remark 1.1. There has also been major effort to estimate the complex Grothendieck constant [14, 8, 30]; the best known upper bound in this case is due to Haagerup [16]. We did not investigate this issue here, partly because for complex scalars there is no clean conjectured exact value of the Grothendieck constant in the spirit of Krivine's conjecture. Nevertheless, it is conceivable that our approach can improve Haagerup's bound on the complex Grothendieck constant as well. We leave this research direction open for future investigations.

In our opinion, the interest in the exact value of K_G does not necessarily arise from the importance of this constant itself, though the reinterpretation of K_G as a fundamental constant in physics and computer science makes it even more interesting to know at least its first few digits. Rather, we believe that it is very interesting to understand the geometric configuration of unit vectors $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ (and matrix a_{ij}) which make the inequality (1) "most difficult". This issue is related to the "rounding problem" in theoretical computer science; see Section 2. With this in mind, Krivine's conjecture corresponds to a natural geometric intuition about the worst spherical configuration for Grothendieck's inequality. This geometric picture has been crystalized and cleanly formulated as an extremal analytic/geometric problem due to the works of Haagerup, König, and Tomczak-Jaegermann. We shall now explain this issue, since one of the main conceptual consequences of Theorem 1.1 is that the geometric picture behind Grothendieck's inequality that was previously believed to be true, is actually false. Along the way, we resolve a conjecture of König [22].

1.1. **König's problem.** One can reformulate Grothendieck's inequality using integral operators (see [22]). Given a measure space (Ω, μ) and a kernel $K \in L_1(\Omega \times \Omega, \mu \times \mu)$, consider the integral operator $T_K : L_\infty(\Omega, \mu) \to L_1(\Omega, \mu)$ induced by K, i.e.,

$$T_K f(x) \stackrel{\mathrm{def}}{=} \int_\Omega f(y) K(x,y) d\mu(y).$$

Grothendieck's inequality asserts that for every $f, g \in L_\infty(\Omega, \mu; \ell_2)$, i.e., two bounded measurable functions with values in Hilbert space,

$$\int_\Omega \int_\Omega K(x,y) \langle f(x), g(y) \rangle d\mu(x) d\mu(y) \leqslant K_G \|T_K\|_{L_\infty(\Omega,\mu) \to L_1(\Omega,\mu)} \|g\|_{L_\infty(\Omega,\mu;\ell_2)} \|f\|_{L_\infty(\Omega,\mu;\ell_2)}. \quad (3)$$

König [22], citing unpublished computations of Haagerup, asserts that the assumption $K_G = \pi / \left(2 \log\left(1+\sqrt{2}\right)\right)$ suggests that the oscillatory Gaussian kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$K(x,y) \stackrel{\mathrm{def}}{=} \exp\left(-\frac{\|x\|_2^2 + \|y\|_2^2}{2}\right) \sin(\langle x, y \rangle) \quad (4)$$

should be extremal for Grothendieck's inequality in the asymptotic sense, i.e., for $n \to \infty$. In the rest of this paper K will always stand for the kernel appearing in (4), and the corresponding bilinear form $B_K : L_\infty(\mathbb{R}^n) \times L_\infty(\mathbb{R}^n) \to \mathbb{R}$ will be given by

$$B_K(f,g) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) K(x,y) dx dy. \quad (5)$$

The above discussion led König to make the following conjecture:

Conjecture 1.2 (König [22]). *Define $f_0 : \mathbb{R}^n \to \{-1, 1\}$ by $f_0(x_1, \ldots, x_n) = \mathrm{sign}(x_1)$. Then $B_K(f, g) \leqslant B_K(f_0, f_0)$ for every $n \in \mathbb{N}$ and every measurable $f, g : \mathbb{R}^n \to \{-1, 1\}$.*

In [22] the following result of König and Tomczak-Jaegermann is proved:

Proposition 1.2 (König and Tomczak-Jaegermann [22]). *A positive answer to Conjecture 1.2 would imply that $K_G = \frac{\pi}{2\log\left(1+\sqrt{2}\right)}$.*

Proposition 1.2 itself can be viewed as motivation for Conjecture 1.2, since it is consistent with Haagerup's work and Krivine's conjecture. But, there are additional reasons why Conjecture 1.2 is natural. First of all, we know due to Lieb's work [25] that general Gaussian kernels, when viewed as operators from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, have only Gaussian maximizers provided p and q satisfy certain conditions. The kernel K does not fit into Lieb's framework, since it is the imaginary part of a Gaussian kernel (the Gaussian Fourier transform) rather than an actual Gaussian kernel, and moreover the range $p = \infty$ and $q = 1$ is not covered by Lieb's theorem. Nevertheless, in light of Lieb's theorem one might expect that maximizers of kernels of this type have a simple structure, which could be viewed as a weak justification of Conjecture 1.2. A much more substantial justification of Conjecture 1.2 is that in [22] König announced an unpublished result that he obtained jointly with Tomczak-Jaegermann asserting that Conjecture 1.2 is true for $n = 1$.

Theorem 1.3. *For every Lebesgue measurable $f, g : \mathbb{R} \to \{-1, 1\}$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\exp\left(-\frac{x^2+y^2}{2}\right)\sin(xy)dxdy \\ \leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{sign}(x)\mathrm{sign}(y)\exp\left(-\frac{x^2+y^2}{2}\right)\sin(xy)dxdy = 2\sqrt{2}\log\left(1+\sqrt{2}\right). \quad (6)$$

Moreover, equality in (6) is attained only when $f(x) = g(x) = \mathrm{sign}(x)$ almost everywhere or $f(x) = g(x) = -\mathrm{sign}(x)$ almost everywhere.

We believe that it is important to have a published proof of Theorem 1.3, and for this reason we prove it in Section 6. Conceivably our proof is similar to the unpublished proof of König and Tomczak-Jaegermann, though they might have found a different explanation of this phenomenon. Since Theorem 1.1 combined with Proposition 1.2 implies that König's conjecture is false, and as we shall see it is false already for $n = 2$, Theorem 1.3 highlights special behavior of the one dimensional case.

Our proof of Theorem 1.1 starts by disproving König's conjecture for $n = 2$. This is done in Section 4. Obtaining an improved upper bound on the Grothendieck constant requires a substantial amount of additional work that uses the counterexample to Conjecture 1.2. This is carried out in Section 5. The failure of König's conjecture shows that the situation is more complicated than originally hoped, and in particular that for $n > 1$ the maximizers of the kernel K have a truly high-dimensional behavior. This more complicated geometric picture highlights the availability of high dimensional rounding schemes that are more sophisticated (and better) than "hyperplane rounding". These issues are discussed in Section 2 and Section 3.

2. Krivine-type rounding schemes and algorithmic implications

Consider the following optimization problem. Given an $m \times n$ matrix $A = (a_{ij})$, compute in polynomial time the value

$$\mathrm{OPT}(A) \stackrel{\mathrm{def}}{=} \max_{\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in \{-1,1\}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \varepsilon_i \delta_j. \tag{7}$$

We refer to [2, 21] for a discussion of the combinatorial significance of this problem. It suffices to say here that it relates to the problem of computing efficiently the Cut Norm of a matrix, which is a subroutine in a variety of applications, starting with the pioneering work of Frieze and Kannan [12]. Special choices of matrices A in (7) lead to specific problems of interest, including efficient construction of Szemerédi partitions [2].

As shown in [2], there exists $\delta_0 > 0$ such that the existence of a polynomial time algorithm that outputs a number that is guaranteed to be within a factor of $1 + \delta_0$ of $\mathrm{OPT}(A)$ would imply that P=NP. But, since the quantity

$$\mathrm{SDP}(A) \stackrel{\mathrm{def}}{=} \max_{x_1,\ldots,x_m,y_1,\ldots,y_n \in S^{m+n-1}} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \langle x_i, y_j \rangle$$

can be computed in polynomial time with arbitrarily good precision (it is a semidefinite program [15]), Grothendieck's inequality tells us that the polynomial time algorithm that outputs the number $\mathrm{SDP}(A)$ is always within a factor of K_G of $\mathrm{OPT}(A)$.

Remarkably, the work of Raghavendra and Steurer [33] shows that K_G has a complexity theoretic interpretation: it is likely that no polynomial time algorithm can approximate $\mathrm{OPT}(A)$ to within a factor smaller than K_G. More precisely, it is shown in [33] that K_G is the Unique Games hardness threshold of the problem of computing $\mathrm{OPT}(A)$. To explain what this means we briefly recall Khot's Unique Games Conjecture [19] (the version described below is equivalent to the original one, as shown in [20]).

Khot's conjecture asserts that for every $\varepsilon > 0$ there exists a prime $p = p(\varepsilon) \in \mathbb{N}$ such that there is no polynomial time algorithm that, given $n \in \mathbb{N}$ and a system of m linear equations in n variables of the form

$$x_i - x_j \equiv c_{ij} \mod p$$

for some $c_{ij} \in \mathbb{N}$, determines whether there exists an assignment of an integer value to each variable x_i such that at least $(1-\varepsilon)m$ of the equations are satisfied, or whether no assignment of such values can satisfy more than εm of the equations (if neither of these possibilities occur, then an arbitrary output is allowed).

The Unique Games Conjecture is by now a common assumption that has numerous applications in computational complexity. We have already seen that there exists a polynomial time algorithm that computes $\mathrm{OPT}(A)$ to within a factor of K_G. The Raghavendra-Steurer theorem says that if there were a polynomial time algorithm ALG that computes $\mathrm{OPT}(A)$ to within a factor $c < K_G$, then the Unique Games Conjecture would be false. This means that there is $\varepsilon = \varepsilon_c \in (0,1)$ such that, for all primes p, Raghavendra and Steurer design an algorithm that makes one call to the algorithm ALG, with at most polynomially many additional Turing machine steps, which successfully solves the problem described above on the satisfiability of linear equations modulo p. Note that Raghavendra and Steurer manage to do this despite the fact that the value of K_G is unknown.

Theorem 1.1 yields the first improved upper bound on the Unique Games hardness threshold of the $\mathrm{OPT}(A)$ computation problem since Krivine's 1977 bound. As we shall see, what hides behind Theorem 1.1 is also a new algorithmic method which is of independent interest. To explain this, note that the above discussion dealt with the problem of computing the *number* $\mathrm{OPT}(A)$. But it is actually of greater interest to find in polynomial time signs $\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in \{-1,1\}$ from among all such 2^{m+n} choices of signs, for which $\sum_{i=1}^m \sum_{j=1}^n a_{ij}\varepsilon_i\delta_j$ is at least a constant multiple $\mathrm{OPT}(A)$. This amounts to a "rounding problem": we need to find a procedure that, given vectors $x_1,\ldots,x_m,y_1,\ldots,y_n \in S^{m+n-1}$, produces signs $\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in \{-1,1\}$ whose existence is ensured by Grothendieck's inequality (1).

Krivine's proof of (2) is based on a clever two-step rounding procedure. We shall now describe a generalization of Krivine's method.

Definition 2.1 (Krivine rounding scheme)**.** *Fix $k \in \mathbb{N}$ and assume that we are given two odd measurable functions $f,g:\mathbb{R}^k \to \{-1,1\}$. Let $G_1,G_2 \in \mathbb{R}^k$ be independent random vectors that are distributed according to the standard Gaussian measure on $\mathbb{R}^k$, i.e., the measure with density $x \mapsto e^{-\|x\|_2^2/2}/(2\pi)^{k/2}$. For $t\in(-1,1)$ define*

$$\begin{aligned} H_{f,g}(t) &\stackrel{\mathrm{def}}{=} \mathbb{E}\left[f\left(\frac{1}{\sqrt{2}}G_1\right) g\left(\frac{t}{\sqrt{2}}G_1+\frac{\sqrt{1-t^2}}{\sqrt{2}}G_2\right)\right]\\ &= \frac{1}{\pi^k(1-t^2)^{k/2}}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} f(x)g(y)\exp\left(\frac{-\|x\|_2^2-\|y\|_2^2+2t\langle x,y\rangle}{1-t^2}\right)dxdy. \end{aligned} \tag{8}$$

Then $H_{f,g}$ extends to an analytic function on the strip $\{z\in\mathbb{C}:\ \Re(z)\in(-1,1)\}$. We shall call $\{f,g\}$ a Krivine rounding scheme if $H_{f,g}$ is invertible on a neighborhood of the origin, and if we consider the Taylor expansion

$$H_{f,g}^{-1}(z) = \sum_{j=0}^{\infty} a_{2j+1} z^{2j+1} \tag{9}$$

then there exists $c=c(f,g)\in(0,\infty)$ satisfying

$$\sum_{j=0}^{\infty} |a_{2j+1}| c^{2j+1} = 1. \tag{10}$$

(Only odd Taylor coefficients appear in (9) since $H_{f,g}$, and therefore also $H_{f,g}^{-1}$, is odd.)

Definition 2.2 (Alternating Krivine rounding scheme)**.** *A Krivine rounding scheme $\{f,g\}$ is called an alternating Krivine rounding scheme if the coefficients $\{a_{2j+1}\}_{j=0}^{\infty}\subseteq\mathbb{R}$ in (9) satisfy $\mathrm{sign}(a_{2j+1})=(-1)^j$ for all $j\in\mathbb{N}\cup\{0\}$. Note that in this case equation (10) becomes $H_{f,g}^{-1}(ic)/i=1$, or*

$$c(f,g)=\frac{H_{f,g}(i)}{i} \stackrel{(4)\wedge(5)\wedge(8)}{=} \frac{B_K(f,g)}{\left(\sqrt{2}\pi\right)^k}. \tag{11}$$

Given a Krivine rounding scheme $f,g:\mathbb{R}^k\to\{-1,1\}$ and $x_1,\ldots,x_m,y_1,\ldots,y_n\in S^{m+n-1}$, the (generalized) Krivine rounding method proceeds via the following two steps.

Step 1 (preprocessing the vectors). Consider the Hilbert space

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} \left(\mathbb{R}^{m+n}\right)^{\otimes(2j+1)}.$$

For $x \in S^{m+n-1}$ we can then define two vectors $I(x), J(x) \in \mathcal{H}$ by

$$I(x) \stackrel{\mathrm{def}}{=} \sum_{j=0}^{\infty} |a_{2j+1}|^{1/2} c^{(2j+1)/2} x^{\otimes(2j+1)} \tag{12}$$

and

$$J(x) \stackrel{\mathrm{def}}{=} \sum_{j=0}^{\infty} \mathrm{sign}(a_{2j+1}) |a_{2j+1}|^{1/2} c^{(2j+1)/2} x^{\otimes(2j+1)}, \tag{13}$$

where $c = c(f,g)$. The choice of c was made in order to ensure that $I(x)$ and $J(x)$ are unit vectors in $\mathcal{H}$. Moreover, the definitions (12) and (13) were made so that the following identity holds:

$$\forall x, y \in S^{m+n-1}, \quad \langle I(x), J(y)\rangle_{\mathcal{H}} = H_{f,g}^{-1}(c\langle x, y\rangle). \tag{14}$$

The preprocessing step of the Krivine rounding method transforms the initial unit vectors $\{x_r\}_{r=1}^m, \{y_s\}_{s=1}^n \subseteq S^{m+n-1}$ to vectors $\{u_r\}_{r=1}^m, \{v_s\}_{s=1}^n \subseteq S^{m+n-1}$ satisfying the identities

$$\forall (r,s) \in \{1,\ldots,m\} \times \{1,\ldots,n\} \quad \langle u_r, v_s\rangle = \langle I(x_r), J(y_s)\rangle_{\mathcal{H}} \stackrel{(14)}{=} H_{f,g}^{-1}(c\langle x_r, y_s\rangle). \tag{15}$$

As explained in [2], these new vectors can be computed efficiently provided $H_{f,g}^{-1}$ can be computed efficiently; this simply amounts to computing a Cholesky decomposition.

Step 2 (random projection). Let $G : \mathbb{R}^{m+n} \to \mathbb{R}^k$ be a random $k \times (m+n)$ matrix whose entries are i.i.d. standard Gaussian random variables. Define random signs $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n \in \{-1, 1\}$ by

$$\forall (r,s) \in \{1,\ldots,m\} \times \{1,\ldots,n\} \quad \sigma_r \stackrel{\mathrm{def}}{=} f\left(\frac{1}{\sqrt{2}} G u_r\right) \quad \text{and} \quad \tau_s \stackrel{\mathrm{def}}{=} g\left(\frac{1}{\sqrt{2}} G v_s\right). \tag{16}$$

Having obtained the random signs $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n \in \{-1,1\}$ as in (16), for every $m \times n$ matrix (a_{rs}) we have

$$\begin{aligned}\max_{\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in \{-1,1\}} \sum_{r=1}^m \sum_{s=1}^n a_{rs}\varepsilon_r\delta_s &\geqslant \mathbb{E}\left[\sum_{r=1}^m \sum_{s=1}^n a_{rs}\sigma_r\tau_s\right] \\ &\stackrel{(\clubsuit)}{=} \mathbb{E}\left[\sum_{r=1}^m \sum_{s=1}^n a_{rs} H_{f,g}\left(\langle u_r, v_s\rangle\right)\right] \stackrel{(15)}{=} c(f,g) \sum_{r=1}^m \sum_{s=1}^n a_{rs}\langle x_r, y_s\rangle,\end{aligned}$$

where $(\clubsuit)$ follows by rotation invariance from (16) and (8). We have thus proved the following corollary, which yields a systematic way to bound the Grothendieck constant from above.

Corollary 2.3. *Assume that $f, g : \mathbb{R}^k \to \{-1,1\}$ is a Krivine rounding scheme. Then*

$$K_G \leqslant \frac{1}{c(f,g)}.$$

Krivine's proof of (2) corresponds to Corollary 2.3 when $k = 1$ and $f(x) = g(x) = \mathrm{sign}(x)$. In this case $\{f, g\}$ is an alternating Krivine rounding scheme with $H_{f,g}(t) = \frac{2}{\pi}\arcsin(t)$ (Grothendieck's identity). By (11) we have $c(f, g) = \frac{2}{\pi i}\arcsin(i) = \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$, so that Corollary 2.3 does indeed correspond to Krivine's bound (2).

One might expect that, since we want to round vectors $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ to signs $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \in \{-1, 1\}$, the best possible Krivine rounding scheme occurs when $k = 1$ and $f(x) = g(x) = \mathrm{sign}(x)$. This is the intuition leading to König's conjecture. The following simple corollary of Theorem 1.3 says that among all *one dimensional* Krivine rounding schemes $f, g : \mathbb{R} \to \{-1, 1\}$ we indeed have $c(f, g) \leqslant c(\mathrm{sign}, \mathrm{sign})$, so it does not pay off to take partitions of $\mathbb{R}$ which are more complicated than the half-line partitions.

Lemma 2.4. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be a Krivine rounding scheme. Then $c(f, g) \leqslant \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$.*

Proof. Denote $c = c(f, g)$ and assume for contradiction that $c > \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$. Let r be the radius of convergence of the power series of $H_{f,g}^{-1}$ given in (9). Due to (10) we know that $r \geqslant c > \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$. Denote

$$\alpha \stackrel{\mathrm{def}}{=} \frac{H_{f,g}(i)}{i} \stackrel{(4)\wedge(5)\wedge(8)}{=} \frac{B_K(f, g)}{\pi\sqrt{2}}. \tag{17}$$

By Theorem 1.3 we have $|\alpha| \leqslant \frac{2}{\pi}\log\left(1+\sqrt{2}\right) < r$, and therefore $H_{f,g}^{-1}$ is well defined at the point $i\alpha \in \mathbb{C}$. Thus,

$$1 \stackrel{(17)}{=} \frac{H_{f,g}^{-1}(i\alpha)}{i} \stackrel{(9)}{=} \sum_{j=0}^{\infty}(-1)^j a_{2j+1}\alpha^{2j+1} \leqslant \sum_{j=0}^{\infty}|a_{2j+1}| \cdot |\alpha|^{2j+1}.$$

By the definition of c in (10) we deduce that $c \leqslant |\alpha| \leqslant \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$, as required. □

The conceptual message behind Theorem 1.1 is that, despite the above satisfactory state of affairs in the one dimensional case, it does pay off to use more complicated higher dimensional partitions. Specifically, our proof of Theorem 1.1 uses the following rounding procedure. Let $c, p \in (0, 1)$ be small enough absolute constants. Given $\{x_r\}_{r=1}^m, \{y_s\}_{s=1}^n \subseteq S^{m+n-1}$, we preprocess them to obtain new vectors $\{u_r = u_r(p, c)\}_{r=1}^m, \{v_s = v_s(p, c)\}_{s=1}^n \subseteq S^{m+n-1}$. Due to certain technical complications, these new vectors are obtained via a procedure that is similar to the preprocessing step (Step 1) described above, but is not identical to it. We refer to Section 5 for a precise description of the preprocessing step that we use (we conjecture that this complication is unnecessary; see Conjecture 5.5). Once the new vectors $\{u_r\}_{r=1}^m, \{v_s\}_{s=1}^n \subseteq S^{m+n-1}$ have been constructed, we take an $2 \times (m+n)$ matrix G with entries that are i.i.d. standard Gaussian random variables, and we consider the random vectors $\{Gu_r = ((Gu_r)_1, (Gu_r)_2)\}_{r=1}^m, \{Gv_s = ((Gv_s)_1, (Gv_s)_2)\}_{s=1}^n \subseteq \mathbb{R}^2$. Having thus obtained new vectors in $\mathbb{R}^2$, with probability $(1-p)$ we "round" our initial vectors to the signs $\{\mathrm{sign}((Gu_r)_2)\}_{r=1}^m, \{\mathrm{sign}((Gv_s)_2)\}_{s=1}^n \subseteq \mathbb{R}$, while with probability p we round x_r to $+1$ if

$$(Gu_r)_2 \geqslant c\left(((Gu_r)_1)^5 - 10((Gu_r)_1)^3 + 15(Gu_r)_1\right). \tag{18}$$

and we round x_r to -1 if

$$(Gu_r)_2 < c\left(((Gu_r)_1)^5 - 10((Gu_r)_1)^3 + 15(Gu_r)_1\right). \tag{19}$$

For concreteness, at this juncture it suffices to describe our rounding procedure without explaining how it was derived — the origin of the fifth degree polynomial appearing in (18) and (19) will become clear in Section 4 and Section 5. The rounding procedure for y_s is identical to (18) and (19), with $(Gv_s)_1, (Gv_s)_2$ replacing $(Gu_r)_1, (Gu_r)_2$, respectively.

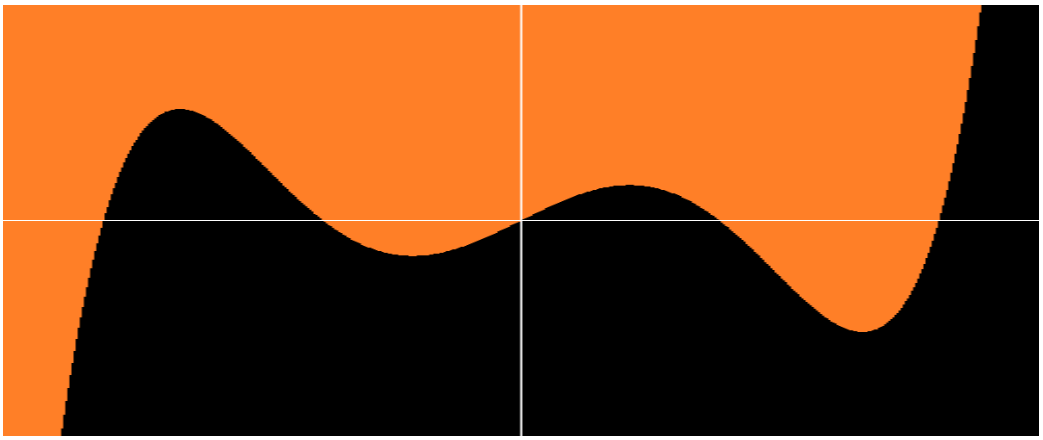

Figure 1. The rounding procedure used in the proof of Theorem 1.1 relies on the partition of $\mathbb{R}^2$ depicted above. After a preprocessing step, high dimensional vectors are projected randomly onto $\mathbb{R}^2$ using a matrix with i.i.d. standard Gaussian entries. With a certain fixed probability, if the projected vector falls above the graph $y = c(x^5 - 10x^3 + 15x)$ then it is assigned the value $+1$, and otherwise it is assigned the value -1.

3. The tiger partition and directions for future research

The partition of the plane described in Figure 1 leads to a proof of Theorem 1.1, but it is not the optimal partition for this purpose. It makes more sense to use the partitions corresponding to maximizers $f_{\max}, g_{\max} : \mathbb{R}^2 \to \{-1, 1\}$ of Krivine's bilinear form B_K as defined in (5), i.e.,

$$\begin{aligned} B_K(f_{\max}, g_{\max}) &= \max_{f,g:\mathbb{R}^2\to\{-1,1\}} B_K(f,g) \\ &= \max_{f,g:\mathbb{R}^2\to\{-1,1\}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)g(y)\exp\left(-\frac{\|x\|_2^2+\|y\|_2^2}{2}\right)\sin(\langle x,y\rangle)dxdy. \quad (20)\end{aligned}$$

A straightforward weak compactness argument shows that the maximum in (20) is indeed attained (see Section 4).

Given $f : \mathbb{R}^2 \to \{-1, 1\}$ define $\sigma(f) : \mathbb{R}^2 \to \{-1, 1\}$ by

$$\sigma(f)(y) \stackrel{\mathrm{def}}{=} \mathrm{sign}\left(\int_{\mathbb{R}^2} f(x)e^{-\|x\|_2^2/2}\sin\left(\langle x,y\rangle\right)dx\right).$$

Then

$$\sigma(f_{\max}) = g_{\max} \quad \text{and} \quad \sigma(g_{\max}) = f_{\max}. \quad (21)$$

Given $f : \mathbb{R}^2 \to \{-1, 1\}$ we can then hope to approach $f_{\max}$ by considering the iterates $\{\sigma^{2j}(f)\}_{j=1}^{\infty}$. If these iterates converge to f_∞ then the pair of functions $\{f_\infty, \sigma(f_\infty)\}$ would

satisfy the equations (21). One can easily check that $\sigma(f_0) = f_0$ when $f_0 : \mathbb{R}^2 \to \{-1, 1\}$ is given by $f_0(x_1, x_2) = \operatorname{sign}(x_2)$. But, we have experimentally applied the above iteration procedure to a variety of initial functions $f \neq f_0$ (both deterministic and random choices), and in all cases the numerical computations suggest that the iterates $\{\sigma^{2j}(f)\}_{j=1}^{\infty}$ converge to the function f_∞ that is depicted in Figure 2 and Figure 3 (the corresponding function $g_\infty = \sigma(f_\infty)$ is different from f_∞, but has a similar structure).

FIGURE 2. The "tiger partition": a depiction of the limiting function f_∞ restricted to the square $[-7, 7] \times [-7, 7] \subseteq \mathbb{R}^2$, based on numerical computations. The two shaded regions correspond to the points where f_∞ takes the values $+1$ and -1.

Question 3.1. *Find an analytic description of the function f_∞ from Figure 2 and Figure 3. Our numerical computations suggest that the iterates $\{\sigma^{2j}(f)\}_{j=1}^{\infty}$ converge to f_∞ for (almost?) all initial data $f : \mathbb{R}^2 \to \{-1, 1\}$ with $f \neq f_0$. Can this statement be made rigorous? If so, is it the case the $\{f_\infty, \sigma(f_\infty)\}$ are maximizers of the bilinear form B_K? We conjecture that the answer to this question is positive.*

Question 3.2. *Analogously to the above planar computations, can one find an analytic description of the maximizers $f_{\max}^{(n)}, g_{\max}^{(n)} : \mathbb{R}^n \to \{-1, 1\}$ of the n-dimensional version of König's bilinear form B_K? If so, does $\{f_{\max}^{(n)}, g_{\max}^{(n)}\}$ form an alternating Krivine rounding scheme (recall Definition 2.2)?*

FIGURE 3. A zoomed-out view of the tiger partition: a depiction of the limiting function f_∞ restricted to the square $[-20,20]\times[-20,20]\subseteq \mathbb{R}^2$, based on numerical computations.

We do not have sufficient data to conjecture whether the answer to Question 3.2 is positive or negative. But, we note that if $\{f^{(n)}_{\max}, g^{(n)}_{\max}\}$ were an alternating Krivine rounding scheme then

$$K_G = \sup_{n\in\mathbb{N}} \frac{\left(\sqrt{2}\pi\right)^n}{B_K\left(f^{(n)}_{\max}, g^{(n)}_{\max}\right)} = \sup_{n\in\mathbb{N}} \frac{\left(\sqrt{2}\pi\right)^n}{\|T_K\|_{L_\infty(\mathbb{R}^n)\to L_1(\mathbb{R}^n)}}. \tag{22}$$

Indeed, assuming that $\{f^{(n)}_{\max}, g^{(n)}_{\max}\}$ is an alternating Krivine rounding scheme the upper bound in (22) follows from Corollary 2.3 and the identity (11). For the reverse inequality in (22) we proceed as in [22]. Using (3) with $f,g:\mathbb{R}^n\to S^{n-1}$ given by $f(x)=g(x)=x/\|x\|_2$, we see that

$$K_G \geqslant \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\frac{\langle x,y\rangle}{\|x\|_2\|y\|_2}dxdy}{\|T_K\|_{L_\infty(\mathbb{R}^n)\to L_1(\mathbb{R}^n)}}, \tag{23}$$

and we conclude that (22) is true since by equation (2.3) in [22] the integral in the numerator of (23) equals $2^{n/2}\pi^n(1-1/n+O(1/n^2))$.

4. A counterexample to König's conjecture

In this section we will make use of several facts on the Hermite polynomials, for which we refer to [3, Sec. 6.1]. We let $\{h_m : \mathbb{R} \to \mathbb{R}\}_{m=0}^{\infty}$ denote the sequence of Hermite polynomials normalized so that they form an orthonormal basis with respect to the measure on $\mathbb{R}$ whose density is $x \mapsto e^{-x^2}$. Explicitly,

$$h_m(x) \stackrel{\mathrm{def}}{=} \frac{(-1)^m}{\sqrt{2^m m! \sqrt{\pi}}} \cdot e^{x^2} \frac{d^m}{dx^m}\left(e^{-x^2}\right), \tag{24}$$

so that

$$\int_{\mathbb{R}} h_m(x) h_k(x) e^{-x^2} dx = \delta_{mk}. \tag{25}$$

For the correctness of the normalization in (24), see equation (6.1.5) in [3].

For reasons that will become clear in Remark 4.1 below, we will consider especially the fifth Hermite polynomial h_5, in which case (24) becomes

$$h_5(x) = \frac{4x^5 - 20x^3 + 15x}{2\sqrt[4]{\pi}\sqrt{15}}.$$

We record for future use the following technical lemma.

Lemma 4.1. *The following identities hold true:*

$$\int_{-\infty}^{\infty} e^{-x^2} h_5(x)^4 dx = \frac{4653}{\sqrt{\pi}}, \tag{26}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x_1^2 + y_1^2}{2}\right) h_5(x_1)^2 h_5(y_1)^2 \cos(x_1 y_1) dx_1 dy_1 = 49\sqrt{2}. \tag{27}$$

Proof. We shall use equation (6.1.7) in [3], which asserts that for every $x, t \in \mathbb{R}$ we have,

$$\sum_{n=0}^{\infty} \frac{h_n(x)}{\sqrt{n!}} t^n = \frac{1}{\sqrt[4]{\pi}} \exp\left(\sqrt{2}tx - \frac{t^2}{2}\right). \tag{28}$$

It follows that for every $x, u_1, u_2, u_3, u_4 \in \mathbb{R}$,

$$\left(\sum_{a=0}^{\infty} \frac{h_a(x)}{\sqrt{a!}} u_1^a\right)\left(\sum_{b=0}^{\infty} \frac{h_b(x)}{\sqrt{b!}} u_2^b\right)\left(\sum_{c=0}^{\infty} \frac{h_c(x)}{\sqrt{c!}} u_3^c\right)\left(\sum_{d=0}^{\infty} \frac{h_d(x)}{\sqrt{d!}} u_4^d\right)$$
$$= \frac{1}{\pi} \exp\left(\sqrt{2}x(u_1 + u_2 + u_3 + u_4) - \frac{u_1^2 + u_2^2 + u_3^2 + u_4^2}{2}\right). \tag{29}$$

Note that for every $A \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} \exp\left(-x^2 + \sqrt{2}Ax\right) dx = e^{A^2/2} \int_{-\infty}^{\infty} \exp\left(-\left(x - \frac{A}{\sqrt{2}}\right)^2\right) dx = \sqrt{\pi} e^{A^2/2}. \tag{30}$$

Hence, if we multiply both sides of (29) by e^{-x^2} and integrate over $x \in \mathbb{R}$, we obtain the following identity:

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty} \frac{u_1^a u_2^b u_3^c u_4^d}{\sqrt{a!b!c!d!}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} h_a(x)h_b(x)h_c(x)h_d(x)dx$$
$$\overset{(29)\wedge(30)}{=} \frac{1}{\sqrt{\pi}} \exp\left(\frac{(u_1+u_2+u_3+u_4)^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2}{2}\right). \quad (31)$$

By equating the coefficients of $u_1^5 u_2^5 u_3^5 u_4^5$ on both sides of (31), we deduce that

$$\frac{1}{14400}\int_{-\infty}^{\infty} e^{-x^2/2} h_5(x)^4 dx = \frac{517}{1600\sqrt{\pi}},$$

implying (26).

To prove (27), starting from (28) we see that for every $x_1, y_1, u_1, u_2, v_1, v_2 \in \mathbb{R}$ we have,

$$\left(\sum_{a=0}^{\infty} \frac{h_a(x_1)}{\sqrt{a!}} u_1^a\right)\left(\sum_{b=0}^{\infty} \frac{h_b(x_1)}{\sqrt{b!}} u_2^b\right)\left(\sum_{c=0}^{\infty} \frac{h_c(y_1)}{\sqrt{c!}} v_1^c\right)\left(\sum_{d=0}^{\infty} \frac{h_d(y_1)}{\sqrt{d!}} v_2^d\right)$$
$$= \frac{1}{\pi}\exp\left(\sqrt{2}x_1(u_1+u_2) + \sqrt{2}y_1(v_1+v_2) - \frac{u_1^2+u_2^2+v_1^2+v_2^2}{2}\right). \quad (32)$$

We next note the following identity: for every $A, B \in \mathbb{R}$ we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(\sqrt{2}Ax + \sqrt{2}By - \frac{x^2+y^2}{2}\right)\cos(xy)dxdy = \sqrt{2}\pi e^{(A^2+B^2)/2}\cos(AB). \quad (33)$$

We will prove (33) in a moment, but let's first assume its validity and see how to complete the proof of (27). If we multiply (32) by $\exp\left(-\frac{x_1^2+x_2^2}{2}\right)\cos(x_1y_1)$, and then integrate with respect to $(x_1, y_1) \in \mathbb{R}^2$, we obtain the following identity:

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty} \frac{u_1^a u_2^b v_1^c v_2^d}{\sqrt{a!b!c!d!}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_a(x_1)h_b(x_1)h_c(y_1)h_d(y_1)\cos(x_1y_1)dx_1dy_1$$
$$\overset{(32)\wedge(33)}{=} \sqrt{2}e^{u_1u_2+v_1v_2}\cos((u_1+u_2)(v_1+v_2)). \quad (34)$$

By equating the coefficients of $u_1^5 u_2^5 v_1^5 v_2^5$ on both sides of (34), we deduce that

$$\frac{1}{14400}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x_1^2+y_1^2}{2}\right) h_5(x_1)^2 h_5(y_1)^2 \cos(x_1y_1)dx_1dy_1 = \frac{49\sqrt{2}}{14400},$$

implying (27).

It remains to prove (33). Let I denote the integral on the right hand side of (33). The change of variable $u = x - \sqrt{2}A$ and $v = y - \sqrt{2}B$ yields the following identity:

$$I = e^{A^2+B^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{u^2+v^2}{2}\right)\cos\left(\left(u+\sqrt{2}A\right)\left(v+\sqrt{2}B\right)\right)dudv. \quad (35)$$

Note that

$$\cos\left(\left(u+\sqrt{2}A\right)\left(v+\sqrt{2}B\right)\right) = \cos\left(u\left(v+\sqrt{2}B\right)\right)\cos\left(\sqrt{2}A\left(v+\sqrt{2}B\right)\right) - \sin\left(u\left(v+\sqrt{2}B\right)\right)\sin\left(\sqrt{2}A\left(v+\sqrt{2}B\right)\right). \quad (36)$$

Since $\int_{-\infty}^{\infty} e^{-u^2/2}\sin\left(u\left(v+\sqrt{2}B\right)\right)du = 0$, it follows that

$$e^{-A^2-B^2}I \overset{(35)\wedge(69)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{u^2+v^2}{2}\right)\cos\left(u\left(v+\sqrt{2}B\right)\right)\cos\left(\sqrt{2}A\left(v+\sqrt{2}B\right)\right)dudv. \quad (37)$$

Since (see equation (6.1.1) in [3]) for all $x_1 \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} e^{-y_1^2/2}\cos(x_1y_1)dy_1 = \sqrt{2\pi}e^{-x_1^2/2}, \quad (38)$$

equation (37) becomes

$$\begin{aligned} e^{-A^2-B^2}I &= \sqrt{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{v^2+\left(v+\sqrt{2}B\right)^2}{2}\right)\cos\left(\sqrt{2}A\left(v+\sqrt{2}B\right)\right)dv \\ &= \sqrt{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\left(\sqrt{2}v+B\right)^2+B^2}{2}\right)\cos\left(\sqrt{2}A\left(v+\sqrt{2}B\right)\right)dv. \end{aligned} \quad (39)$$

The change of variable $w=\sqrt{2}c+B$ in (39) gives,

$$\begin{aligned} e^{-A^2-B^2}I &= \sqrt{\pi}e^{-B^2/2}\int_{-\infty}^{\infty} e^{-w^2/2}\cos\left(A\left(w-B\right)\right)dw \\ &= \sqrt{\pi}e^{-B^2/2}\int_{-\infty}^{\infty} e^{-w^2/2}\left(\cos\left(Aw\right)\cos(AB)+\sin(Aw)\sin(AB)\right)dw \\ &\overset{(38)}{=} \sqrt{2}\pi e^{-(A^2+B^2)/2}\cos(AB). \end{aligned}$$

This conclude the proof of (33), and therefore the proof of Lemma 4.1 is complete. □

For $\eta \in (0,1)$ let $f_\eta : \mathbb{R}^2 \to \{-1,1\}$ be given by

$$f_\eta(x_1,x_2) \overset{\text{def}}{=} \begin{cases} 1 & x_2 \geqslant \eta h_5(x_1), \\ -1 & x_2 < \eta h_5(x_1). \end{cases} \quad (40)$$

Note that since h_5 is odd, so is f_η (almost surely). For $z \in \mathbb{C}$ with $|\Re(z)|<1$ we define

$$H_\eta(z) \overset{\text{def}}{=} \frac{1}{2\pi(1-z^2)}\int_{\mathbb{R}^2\times\mathbb{R}^2} f_\eta(x)f_\eta(y)\exp\left(\frac{-\|x\|_2^2-\|y\|_2^2+2z\langle x,y\rangle}{1-z^2}\right)dxdy. \quad (41)$$

Lemma 4.2. *H_η is analytic on the strip*

$$\mathbb{S} \overset{\text{def}}{=} \{z\in\mathbb{C}: \ |\Re(z)|<1\}. \quad (42)$$

Moreover, for all $a+bi \in \mathbb{S}$ we have

$$|H_\eta(a+bi)| \leqslant \frac{\pi\left((1+a)^2+b^2\right)\left((1-a)^2+b^2\right)}{2(1-a^2)\sqrt{(1-a^2)^2+b^4+2(1+a^2)b^2}}. \quad (43)$$

Proof. Assume that $a \in (-1,1)$ and $b \in \mathbb{R}$, and write $z = a + bi$. Then

$$\begin{aligned}
&|H_\eta(z)| \\
&\leqslant \frac{1}{2\pi|1-z^2|}\int_{\mathbb{R}^2\times\mathbb{R}^2} \exp\left(-\Re\left(\frac{1}{1-z^2}\right)\left(\|x\|_2^2+\|y\|_2^2\right)+2\Re\left(\frac{z}{1-z^2}\right)\langle x,y\rangle\right)dxdy \\
&= \frac{1}{2\pi|1-z^2|}\int_{\mathbb{R}^2\times\mathbb{R}^2} \exp\left(-\frac{1}{2}\Re\left(\frac{1}{1+z}\right)\|x+y\|_2^2-\frac{1}{2}\Re\left(\frac{1}{1-z}\right)\|x-y\|_2^2\right)dxdy \\
&= \frac{1}{8\pi|1-z^2|}\int_{\mathbb{R}^2\times\mathbb{R}^2} \exp\left(-\frac{1}{2}\Re\left(\frac{1}{1+z}\right)\|u\|_2^2-\frac{1}{2}\Re\left(\frac{1}{1-z}\right)\|v\|_2^2\right)dudv \qquad (44) \\
&= \frac{1}{8\pi|1-z^2|}\cdot\frac{(2\pi)^2}{\Re\left(\frac{1}{1-z}\right)\Re\left(\frac{1}{1+z}\right)} \\
&= \frac{\pi\left((1+a)^2+b^2\right)\left((1-a)^2+b^2\right)}{2(1-a^2)\sqrt{(1-a^2)^2+b^4+2(1+a^2)b^2}}. \qquad (45)
\end{aligned}$$

where in (44) we used the change of variable $x+y=u$ and $x-y=v$, whose Jacobian equals $\frac{1}{4}$. Since the integral defining H_η is absolutely convergent on $\mathbb{S}$, the claim follows. $\square$

Lemma 4.3. *For every $z \in \mathbb{C}$ with $|\Re(z)| < 1$ we have $H_0(z) = \arcsin(z)$.*

Proof. It suffices to prove this for $z \in (0,1)$. Writing $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
H_0(z) = &\left(\frac{1}{2\pi(1-z^2)}\int_{\mathbb{R}\times\mathbb{R}}\operatorname{sign}(x_2)\operatorname{sign}(y_2)\exp\left(\frac{-x_2^2-y_2^2+2zx_2y_2}{1-z^2}\right)dx_1dy_1\right) \\
&\cdot\left(\int_{\mathbb{R}\times\mathbb{R}}\exp\left(\frac{-x_1^2-y_1^2+2zx_1y_1}{1-z^2}\right)dx_2dy_2\right) \\
= &\frac{1-z^2}{2\pi}\left(\int_{\mathbb{R}\times\mathbb{R}}\operatorname{sign}(u)\operatorname{sign}(v)e^{-u^2-v^2+2zuv}dudv\right)\left(\int_{\mathbb{R}\times\mathbb{R}}e^{-(u-zv)^2-(1-z^2)v^2}dudv\right). \qquad (46)
\end{aligned}$$

Now,

$$\int_{\mathbb{R}\times\mathbb{R}} e^{-(u-zv)^2-(1-z^2)v^2}dudv = \left(\int_{\mathbb{R}}e^{-w^2}dw\right)\left(\int_{\mathbb{R}}e^{-(1-z^2)v^2}dv\right) = \frac{\pi}{\sqrt{1-z^2}}. \qquad (47)$$

Define

$$\psi(z) = \int_{\mathbb{R}\times\mathbb{R}}\operatorname{sign}(u)\operatorname{sign}(v)e^{-u^2-v^2+2zuv}dudv = 2\int_0^\infty\int_0^\infty e^{-u^2-v^2}\left(e^{2zuv}-e^{-2zuv}\right)dudv.$$

Passing to the polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$, and then making the change of variable $r = \sqrt{s}$, we see that

$$\begin{aligned}\psi(z) &= 2\int_0^{\frac{\pi}{2}}\int_0^{\infty} re^{-r^2}\left(e^{zr^2\sin(2\theta)} - e^{-zr^2\sin(2\theta)}\right)drd\theta \\ &= \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-s}\left(e^{sz\sin(2\theta)} - e^{-sz\sin(2\theta)}\right)dsd\theta \\ &= \int_0^{\frac{\pi}{2}}\left(\frac{1}{1-z\sin(2\theta)} - \frac{1}{1+z\sin(2\theta)}\right)d\theta \\ &= \int_0^{\frac{\pi}{2}}\frac{2z\sin(2\theta)}{1-z^2\sin^2(2\theta)}d\theta.\end{aligned}$$

Due to the identity

$$\frac{d}{d\theta}\left(\frac{-1}{\sqrt{1-z^2}}\arcsin\left(\frac{z\cos(2\theta)}{\sqrt{1-z^2\sin^2(2\theta)}}\right)\right) = \frac{2z\sin(2\theta)}{1-z^2\sin^2(2\theta)},$$

we conclude that

$$\psi(z) = \frac{2\arcsin(z)}{\sqrt{1-z^2}}. \tag{48}$$

Lemma 4.3 now follows from substituting (47) and (48) into (46). □

Theorem 4.4. *There exists $\eta_0 > 0$ such that for all $\eta \in (0,\eta_0)$ we have*

$$\frac{H_\eta(i)}{i} \in \left(\log\left(1+\sqrt{2}\right), \infty\right).$$

Theorem 4.4 implies that the answer to König's problem is negative. Indeed,

$$\frac{H_\eta(i)}{i} \overset{(41)}{=} \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} f_\eta(x)f_\eta(y)\exp\left(-\frac{\|x\|_2^2+\|y\|_2^2}{2}\right)\sin\left(\langle x,y\rangle\right)dxdy = \frac{B_K(f_\eta,f_\eta)}{4\pi}.$$

Since $\arcsin(i) = i\log\left(1+\sqrt{2}\right)$, it follows from Lemma 4.3 and Theorem 4.4 that for every $\eta \in (0,\eta_0)$ we have $B_K(f_\eta,f_\eta) > B_K(f_0,f_0)$. Since $f_0(x_1,x_2) = \operatorname{sign}(x_2)$, the claimed negative answer to König's problem follows.

Proof of Theorem 4.4. Define $\varphi(\eta) = 4\pi H_\eta(i)/i$. The required result will follow once we prove that $\varphi(\eta) = \varphi(0) + 1600\sqrt{2}\eta^4 + O\left(\eta^6\right)$ as $\eta \to 0$. To this end, since φ is even, it suffices to show that $\varphi''(0) = 0$ and $\varphi''''(0) = 38400\sqrt{2}$.

Since h_5 is odd we have,

$$\begin{aligned}&\varphi(\eta) \\ &= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\eta h_5(x_1)}^{\infty}\int_{\eta h_5(y_1)}^{\infty}\exp\left(-\frac{x_1^2+x_2^2+y_1^2+y_2^2}{2}\right)\sin(x_1y_1+x_2y_2)dx_2dy_2dx_1dy_1.\end{aligned}$$

Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned}&F(u_1,u_2) \\ &\stackrel{\mathrm{def}}{=} 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{u_1 h_5(x_1)}^{\infty}\int_{u_2 h_5(y_1)}^{\infty}\exp\left(-\frac{x_1^2+x_2^2+y_1^2+y_2^2}{2}\right)\sin(x_1y_1+x_2y_2)dx_2dy_2dx_1dy_1,\end{aligned}$$

so that $\varphi(\eta) = F(\eta, \eta)$. Since F is symmetric, i.e., $F(u_1, u_2) = F(u_2, u_1)$, it follows that

$$\varphi''(0) = 2\frac{\partial^2 F}{\partial u_1^2}(0,0) + 2\frac{\partial^2 F}{\partial u_1 \partial u_2}(0,0), \tag{49}$$

and

$$\varphi''''(0) = 2\frac{\partial^4 F}{\partial u_1^4}(0,0) + 8\frac{\partial^4 F}{\partial u_1 \partial u_2^3}(0,0) + 6\frac{\partial^4 F}{\partial u_1^2 \partial u_2^2}(0,0). \tag{50}$$

Now,

$$\begin{aligned}\frac{\partial F}{\partial u_1}(u_1, u_2) = -4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{u_2 h_5(y_1)}^{\infty} \exp\left(-\frac{x_1^2 + u_1^2 h_5(x_1)^2 + y_1^2 + y_2^2}{2}\right)& \\ \cdot\, h_5(x_1)\sin\left(x_1 y_1 + u_1 h_5(x_1) y_2\right) dy_2 dx_1 dy_1.&\end{aligned} \tag{51}$$

By differentiation of (51) under the integral with respect to u_1, we see that

$$\begin{aligned}\frac{\partial^2 F}{\partial u_1^2}(0,0) &= -4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \exp\left(-\frac{x_1^2 + y_1^2 + y_2^2}{2}\right) h_5(x_1)^2 y_2 \cos(x_1 y_1) dy_2 dx_1 dy_1 \\ &= -4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x_1^2 + y_1^2}{2}\right) h_5(x_1)^2 \cos(x_1 y_1) dx_1 dy_1 \\ &\overset{(38)}{=} -4\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-x_1^2} h_5(x_1)^2 dx_1 \overset{(25)}{=} -4\sqrt{2\pi}.\end{aligned} \tag{52}$$

By differentiation of (51) with respect to u_2 we see that

$$\begin{aligned}\frac{\partial^2 F}{\partial u_1 \partial u_2}(u_1, u_2) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x_1^2 + u_1^2 h_5(x_1)^2 + y_1^2 + u_2^2 h_5(y_1)^2}{2}\right)& \\ \cdot\, h_5(x_1) h_5(y_1) \sin\left(x_1 y_1 + u_1 u_2 h_5(x_1) h_5(x_2)\right) dx_1 dy_1.&\end{aligned} \tag{53}$$

Hence,

$$\frac{\partial^2 F}{\partial u_1 \partial u_2}(0,0) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x_1^2 + y_1^2}{2}\right) h_5(x_1) h_5(y_1) \sin\left(x_1 y_1\right) dx_1 dy_1. \tag{54}$$

By equation (6.1.15) of [3] we have

$$\int_{-\infty}^{\infty} e^{-y_1^2/2} h_5(y_1) \sin(x_1 y_1) dy_1 = \sqrt{2\pi} e^{-x_1^2/2} h_5(x_1). \tag{55}$$

Hence,

$$\frac{\partial^2 F}{\partial u_1 \partial u_2}(0,0) \overset{(54)\wedge(55)}{=} 4\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-x_1^2} h_5(x_1)^2 dx_1 \overset{(25)}{=} 4\sqrt{2\pi}. \tag{56}$$

By substituting (52) and (56) into (49), we see that $\varphi''(0) = 0$.

We shall now proceed to compute $\varphi''''(0)$, using the identity (50). By differentiation of (51) under the integral with respect to u_1 three times, we see that

$$\begin{aligned}
&\frac{\partial^4 F}{\partial u_1^4}(0,0)\\
&= 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\exp\left(-\frac{x_1^2+y_1^2+y_2^2}{2}\right)\cos(x_1y_1)h_5(x_1)^4\left(y_2+\frac{y_2^3}{3}\right)dy_2dx_1dy_1\\
&= 20\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x_1^2+y_1^2}{2}\right)\cos(x_1y_1)h_5(x_1)^4dx_1dy_1\\
&\overset{(38)}{=} 20\sqrt{2\pi}\int_{-\infty}^{\infty}e^{-x_1^2}h_5(x_1)^4dx_1 \overset{(26)}{=} 93060\sqrt{2}. \qquad (57)
\end{aligned}$$

Now, by differentiation of (53) under the integral with respect to u_2 twice, we obtain the identity

$$\begin{aligned}
\frac{\partial^4 F}{\partial u_1\partial u_2^3}(0,0) &= -4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x_1^2+y_1^2}{2}\right)h_5(y_1)^3h_5(x_1)\sin(x_1y_1)dx_1dy_1\\
&\overset{(55)}{=} -4\sqrt{2\pi}\int_{-\infty}^{\infty}e^{-y_1^2}h_5(y_1)^4dy_1 \overset{(26)}{=} -18612\sqrt{2}. \qquad (58)
\end{aligned}$$

Finally, by differentiation of (53) under the integral once with respect to u_1 and once with respect to u_2, we see that

$$\frac{\partial^4 F}{\partial u_1^2\partial u_2^2}(0,0) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x_1^2+y_1^2}{2}\right)h_5(x_1)^2h_5(y_1)^2\cos(x_1y_1)dx_1dy_1 \overset{(27)}{=} 196\sqrt{2}. \qquad (59)$$

Hence,

$$\varphi''''(0) \overset{(50)\wedge(57)\wedge(58)\wedge(59)}{=} 38400\sqrt{2}.$$

□

Remark 4.1. Clearly, we did not arrive at the above proof by guessing that the fifth Hermite polynomial h_5 is the correct choice in (40). We arrived at this choice as the simplest member of a general family of ways to perturb the function $(x_1,x_2)\mapsto \mathrm{sign}(x_2)$. Since carrying out the analysis of this perturbation procedure in full generality is quite tedious, we chose to present the shorter proof above. Neverthless, we would like to explain here how we arrived at the choice of h_5 in (40).

Fix two odd functions $\alpha,\beta:\mathbb{R}\to\mathbb{R}$, and write their Hermite expansion as

$$\alpha(x)=\sum_{k=0}^{\infty}a_kh_{2k+1}(x) \quad\text{and}\quad \beta(x)=\sum_{k=0}^{\infty}b_kh_{2k+1}(x).$$

For $\eta>0$ define $f_\eta,g_\eta:\mathbb{R}^2\to\{-1,1\}$ by

$$f_\eta(x_1,x_2)=\begin{cases}1 & x_2\geqslant \eta\alpha(x_1),\\ -1 & x_2<\eta\alpha(x_1),\end{cases} \quad\text{and}\quad g_\eta(x_1,x_2)=\begin{cases}1 & x_2\geqslant \eta\beta(x_1),\\ -1 & x_2<\eta\beta(x_1).\end{cases} \qquad (60)$$

For $z\in\mathbb{C}$ with $|\Re(z)|<1$ define

$$H_\eta(z)=\frac{1}{2\pi(1-z^2)}\int_{\mathbb{R}^2\times\mathbb{R}^2}f_\eta(x)g_\eta(y)\exp\left(\frac{-\|x\|_2^2-\|y\|_2^2+2z\langle x,y\rangle}{1-z^2}\right)dxdy.$$

Thus, our final choice (41) corresponds to $\alpha = \beta = h_5$.

As in the proof of Theorem 4.4, define $\varphi(\eta) = 4\pi H_\eta(i)/i$. In order to show that we have $\varphi(\eta) > \varphi(0)$ for small enough $\eta \in (0,1)$, we start by computing $\varphi''(0)$, which turns out to be given by the following formula:

$$\varphi''(0) = -4\sqrt{2\pi}\sum_{k=0}^{\infty}\left(a_k - (-1)^k b_k\right)^2 \leqslant 0. \tag{61}$$

Therefore, since φ is odd and hence its odd order derrivatives at 0 vanish, in order for us to have a chance to prove that $\varphi(\eta) > \varphi(0)$ for small enough $\eta \in (0,1)$, we must have $\varphi''(0) = 0$. Due to (61), in terms of Hermite coefficients this forces the constraints

$$\forall k \in \mathbb{N}\cup\{0\}, \quad a_k = (-1)^k b_k. \tag{62}$$

We can therefore at best hope that $\varphi(\eta)$ is greater than $\varphi(0)$ by a fourth order term, and we need to compute $\varphi''''(0)$. This is possible to do, using some identities from [4]. Denote for $a, b, c, d, k \in \mathbb{N}\cup\{0\}$,

$$L_k(a,b,c,d) \stackrel{\text{def}}{=} \frac{\sqrt{2(2a+1)!(2b+1)!(2c+1)!(2d+1)!}}{(2k)!(a+b+1-k)!(c+d+1-k)!}\cdot\binom{2k}{a-b+k}\binom{2k}{c-d+k},$$

where we use the convention that $L_k(a,b,c,d) = 0$ whenever there is a negative number in the above factorials or binomial coefficients. A somewhat tedious computation that uses results from [4] shows that if the constraint (62) is satisfied then

$$\varphi''''(0) = 8\sum_{\substack{(k,p,q,r,s)\in\mathbb{N}\cup\{0\}\\ p+q+r+s \text{ is even}}} a_p a_q a_r a_s L_k(p,q,r,s)\left(1 + 3(-1)^{k+r+s}\right).$$

If, for simplicity, we want to make the choice $\alpha = \beta$, the simplest solution of the constraints (62) comes from taking $\alpha = \beta = h_5$ ($\alpha = \beta = h_1$ won't work since then one can check that $\varphi(\eta) = \varphi(0)$ for all η). Note, however, that $\alpha = -\beta = h_3$ would work here too.

5. Proof that $K_G < \frac{\pi}{2\log\left(1+\sqrt{2}\right)}$

We will fix from now on some $\eta \in (0,\eta_0)$, where η_0 is as in Theorem 4.4. For $p \in [0,1]$ define

$$F_p \stackrel{\text{def}}{=} (1-p)H_0 + pH_\eta,$$

where H_η is as in (41). In what follows we will denote the unit disc in $\mathbb{C}$ by

$$\mathbb{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \ |z| < 1\}.$$

Theorem 5.1. *The exists $p_0 > 0$ such that for all $p \in (0,p_0)$ we have $F_p(\mathbb{S}) \supseteq \frac{9}{10}\mathbb{D}$ and F_p^{-1} is well defined and analytic on $\frac{9}{10}\mathbb{D}$. Moreover, if we write $F_p^{-1}(z) = \sum_{k=1}^{\infty} a_k(p)z^k$ then there exists $\gamma = \gamma_p \in [0,\infty)$ satisfying*

$$\sum_{k=1}^{\infty}|a_k(p)|\gamma^k = 1, \tag{63}$$

and

$$\gamma > \log\left(1+\sqrt{2}\right) = 0.88137... \tag{64}$$

Assuming Theorem 5.1 for the moment, we will now deduce Theorem 1.1.

Proof of Theorem 1.1. Fix $p \in (0, p_0)$ and let $\gamma > 0$ be the constant from Theorem 5.1. Due to (63), $\sum_{k=1}^{\infty} a_k(p)\gamma^k$ converges absolutely, and therefore F_p^{-1} is analytic and well defined on $\gamma\mathbb{D}$. For small enough p some of the coefficients $\{a_k(p)\}_{k=1}^{\infty}$ are negative (since the third Taylor coefficient of $H_0^{-1}(z) = \sin z$ is negative), implying that for every $r \in [0, 1]$ we have

$$F_p^{-1}(r\gamma) = \sum_{k=1}^{\infty} a_k(p) r^k \gamma^k \in (-1, 1) \subseteq \mathbb{S}. \tag{65}$$

Let $\mathcal{H}$ be a Hilbert space. Define two mappings

$$L_p, R_p : \mathcal{H} \to \bigoplus_{k=1}^{\infty} \mathcal{H}^{\otimes k} \stackrel{\text{def}}{=} \mathcal{K}$$

by

$$L_p(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \sqrt{|a_k(p)|}\gamma^{k/2} x^{\otimes k} \quad \text{and} \quad R_p(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \operatorname{sign}(a_k(p))\sqrt{|a_k(p)|}\gamma^{k/2} x^{\otimes k}.$$

By (63), if $\|x\|_{\mathcal{H}} = 1$ then $\|L_p(x)\|_{\mathcal{K}} = \|R_p(x)\|_{\mathcal{K}} = 1$. Moreover, if $\|x\|_{\mathcal{H}} = \|y\|_{\mathcal{H}} = 1$ then

$$\langle L_p(x), R_p(y)\rangle = \sum_{k=1}^{\infty} a_k(p)\gamma^k \langle x, y\rangle^k = F_p^{-1}(\gamma\langle x, y\rangle) \stackrel{(65)}{\in} \mathbb{S}. \tag{66}$$

For $N \in \mathbb{N}$ let $G : \mathbb{R}^N \to \mathbb{R}^2$ be a $2 \times N$ random matrix with i.i.d. standard Gaussian entries. Let $g_1, g_2 \in \mathbb{R}^2$ be the first two columns of G (i.e., g_1, g_2 are i.i.d. standard two dimensional Gaussian vectors). If $x, y \in \mathbb{R}^N$ are unit vectors satisfying $\langle x, y\rangle \in \mathbb{S}$ then by rotation invariance we have

$$\begin{aligned}
&\mathbb{E}\left[f_\eta\left(\frac{1}{\sqrt{2}}Gx\right) f_\eta\left(\frac{1}{\sqrt{2}}Gy\right)\right] = \mathbb{E}\left[f_\eta\left(\frac{g_1}{\sqrt{2}}\right) f_\eta\left(\langle x, y\rangle\frac{g_1}{\sqrt{2}} + \sqrt{1-\langle x, y\rangle^2}\frac{g_2}{\sqrt{2}}\right)\right] \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} f_\eta\left(\frac{u}{\sqrt{2}}\right) f_\eta\left(\langle x, y\rangle\frac{u}{\sqrt{2}} + \sqrt{1-\langle x, y\rangle^2}\frac{v}{\sqrt{2}}\right)\exp\left(-\frac{\|u\|_2^2 + \|v\|_2^2}{2}\right) du dv \\
&= \frac{2}{\pi} H_\eta(\langle x, y\rangle),
\end{aligned} \tag{67}$$

where we made the change of variable $u = \sqrt{2}u'$ and $v = \left(\sqrt{2}v' - \sqrt{2}\langle x, y\rangle u'\right)/\sqrt{1-\langle x, y\rangle^2}$, whose Jacobian is $4/(1-\langle x, y\rangle^2)$.

Fix an $m \times n$ matrix $A = (a_{ij})$ and let $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathcal{H}$ be unit vectors satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i, y_j\rangle = M \stackrel{\text{def}}{=} \max_{u_1,\ldots,u_m,v_1,\ldots,v_n \in S_{\mathcal{H}}} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle u_i, v_j\rangle, \tag{68}$$

where $S_{\mathcal{H}}$ denotes the unit sphere of $\mathcal{H}$. Consider the unit vectors $\{L_p(x_i)\}_{i=1}^m \cup \{R_p(y_j)\}_{j=1}^n$, which we can think of as residing in $\mathbb{R}^N$ for $N = m+n$. By (66) we have $\langle L_p(x_i), R_p(y_j)\rangle \in \mathbb{S}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, so that we may use the identity (67) for these

vectors. Let λ be a random variable satisfying $\Pr[\lambda = 1] = p$, $\Pr[\lambda = 0] = 1 - p$. Assume that λ is independent of G. Define random variables $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \in \{-1, 1\}$ by

$$\varepsilon_i = (1-\lambda) f_0 \left(\frac{1}{\sqrt{2}} G L_p(x_i) \right) + \lambda f_\eta \left(\frac{1}{\sqrt{2}} G L_p(x_i) \right)$$

and

$$\delta_j = (1-\lambda) f_0 \left(\frac{1}{\sqrt{2}} G R_p(y_j) \right) + \lambda f_\eta \left(\frac{1}{\sqrt{2}} G R_p(y_j) \right).$$

Then,

$$\begin{aligned}
\max_{\sigma_1,\ldots,\sigma_m,\tau_1,\ldots,\tau_n \in \{-1,1\}} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \sigma_i \tau_j &\geqslant \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^n a_{ij} \varepsilon_i \delta_j \right] \\
&\overset{(67)}{=} \frac{2}{\pi} \sum_{i=1}^m \sum_{j=1}^n a_{ij} \Big((1-p) H_0\big(\langle L_p(x_i), R_p(y_j)\rangle\big) + p H_\eta\big(\langle L_p(x_i), R_p(y_j)\rangle\big) \Big) \\
&= \frac{2}{\pi} \sum_{i=1}^m \sum_{j=1}^n a_{ij} F_p\big(\langle L_p(x_i), R_p(y_j)\rangle\big) \\
&\overset{(66)}{=} \frac{2}{\pi} \sum_{i=1}^m \sum_{j=1}^n a_{ij} F_p\left(F_p^{-1} \gamma \langle x_i, y_j\rangle \right) \overset{(68)}{=} \frac{2\gamma}{\pi} M.
\end{aligned}$$

This gives the bound $K_G \leqslant \frac{\pi}{2\gamma} \overset{(64)}{<} \frac{\pi}{2\log\left(1+\sqrt{2}\right)}$, as required. ◻

Our goal from now on will be to prove Theorem 5.1.

Lemma 5.2. *H_0 is one-to-one on $\mathbb{S}$ and $H_0(\mathbb{S}) \supseteq \mathbb{D}$.*

Proof. The fact that H_0 is one-to-one on $\mathbb{S}$ is a consequence of Lemma 4.3. To show that $H_0(\mathbb{S}) \supseteq \mathbb{D}$ we need to prove that if $a, b \in \mathbb{R}$ and $a^2 + b^2 < 1$ then $|\Re\left(\sin\left(a + bi\right)\right)| < 1$. Now,

$$|\Re\left(\sin\left(a + bi\right)\right)| = \frac{e^b + e^{-b}}{2} |\sin a|. \tag{69}$$

Using the inequality $|\sin a| \leqslant |a|$, we see that it suffices to show that for all $x \in (0,1)$ we have

$$\frac{e^x + e^{-x}}{2} \sqrt{1 - x^2} < 1 \tag{70}$$

By Taylor's formula we know that there exists $y \in [0, x]$ such that

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{24} \cdot \frac{e^y + e^{-y}}{2} \leqslant 1 + \frac{x^2}{2} + \frac{x^4}{24} \cdot \frac{e + e^{-1}}{2} < 1 + \frac{x^2}{2} + \frac{x^4}{12}. \tag{71}$$

Note that

$$\left(1 + \frac{x^2}{2} + \frac{x^4}{12}\right)^2 \left(1 - x^2\right) = 1 - \frac{7x^4}{12} - \frac{x^6}{3} - \frac{11x^8}{144} - \frac{x^{10}}{144} < 1,$$

which together with (71) implies (70). ◻

Remark 5.1. A more careful analysis of the expression (69) shows that there exists $\varepsilon_0 > 0$ (e.g., $\varepsilon_0 = 0.05$ works) such that $H_0(\mathbb{S}) \supseteq (1 + \varepsilon_0)\mathbb{D}$. Since we will not need this stronger fact here, we included the above simpler proof of a weaker statement.

Lemma 5.3. *For every $r \in (0,1)$ there exists $p_r \in (0,1)$ and a bounded open subset $\Omega_r \subseteq \mathbb{S}$ with $\overline{\Omega_r} \subseteq \mathbb{S}$ such that for all $p \in (0, p_r)$ the function F_p is one-to-one on Ω_r and $F_p(\Omega_r) = r\mathbb{D}$. Thus F_p^{-1} is well defined and analytic on $r\mathbb{D}$.*

Proof. For $n \in \mathbb{N}$ consider the set

$$E_n = \left\{ z \in \mathbb{C} : \ |\Re(z)| < 1 - \frac{1}{n} \ \wedge \ |\Im(z)| < n \right\}.$$

Using Lemma 5.2, fix a large enough $n \in \mathbb{N}$ so that $H_0(E_n) \supseteq r\overline{\mathbb{D}}$. The bound (43) implies that there exists $M > 0$ such that $|H_\eta(z)| \leqslant M$ for all $\eta > 0$ and $z \in \partial E_{n+1}$. By Lemma 5.2, H_0 takes a value $\zeta \in r\overline{\mathbb{D}}$ exactly once on E_{n+1}, and this occurs at some point in E_n. Hence,

$$m \stackrel{\text{def}}{=} \min_{\substack{\zeta \in r\mathbb{D} \\ z \in \partial E_{n+1}}} |H_0(z) - \zeta| > 0.$$

Define $p_r = m/(2M)$.

Fix $\zeta \in r\mathbb{D}$. If $p \in (0, p_r)$ then for every $z \in \partial E_{n+1}$ we have

$$|p\,(H_\eta(z) - H_0(z))| < \frac{m}{2M}\,(|H_\eta(z)| + |H_0(z)|) \leqslant m \leqslant |H_0(z) - \zeta|\,.$$

Rouché's theorem now implies that the number of zeros of $H_0 - \zeta$ in E_{n+1} is the same as the number of zeros of $H_0 - \zeta + p\,(H_\eta - H_0) = F_p - \zeta$ in E_{n+1}. Hence F_p takes the value ζ exactly once in E_{n+1}. Since ζ was an arbitrary point in $r\mathbb{D}$, we can define $\Omega_r = F_p^{-1}(r\mathbb{D})$. $\square$

Lemma 5.4. *For every $r \in (0,1)$ there exists $C_r \in (0, \infty)$ such that, using the notation of Lemma 5.3, for every $p \in (0, p_r)$ and $z \in r\mathbb{D}$ we have*

$$\left| F_p^{-1}(z) - \sin z - p\,(z - H_\eta(\sin z)) \cos z \right| \leqslant C_r p^2. \tag{72}$$

Proof. Note that

$$z = F_p\left(F_p^{-1}(z)\right) = (1-p) H_0\left(F_p^{-1}(z)\right) + p H_\eta\left(F_p^{-1}(z)\right). \tag{73}$$

By differentiating (73) with respect to p, we see that

$$\begin{aligned} 0 &= H_\eta\left(F_p^{-1}(z)\right) - H_0\left(F_p^{-1}(z)\right) \\ &\quad + \left(\frac{d}{dp} F_p^{-1}(z)\right)\left((1-p)\frac{dH_0}{dz}\left(F_p^{-1}(z)\right) + p\frac{dH_\eta}{dz}\left(F_p^{-1}(z)\right)\right) \\ &= H_\eta\left(F_p^{-1}(z)\right) - H_0\left(F_p^{-1}(z)\right) + \left(\frac{d}{dp} F_p^{-1}(z)\right)\frac{dF_p}{dz}\left(F_p^{-1}(z)\right) \\ &= H_\eta\left(F_p^{-1}(z)\right) - H_0\left(F_p^{-1}(z)\right) + \frac{\frac{d}{dp}F_p^{-1}(z)}{\frac{d}{dz}\left(F_p^{-1}(z)\right)}. \end{aligned}$$

Hence,

$$\frac{d}{dp} F_p^{-1}(z) = \left(H_0\left(F_p^{-1}(z)\right) - H_\eta\left(F_p^{-1}(z)\right)\right)\frac{d}{dz}\left(F_p^{-1}(z)\right). \tag{74}$$

If we now differentiate (74) with respect to p, while using (74) whenever the term $\frac{d}{dp}F_p^{-1}(z)$ appears, we obtain the following identity.

$$\begin{aligned} &\frac{d^2}{dp^2}F_p^{-1}(z) \\ &= \left[\frac{dH_0}{dz}\left(F_p^{-1}(z)\right) - \frac{dH_\eta}{dz}\left(F_p^{-1}(z)\right) + \left(H_0\left(F_p^{-1}(z)\right) - H_\eta\left(F_p^{-1}(z)\right)\right)\frac{d^2}{dz^2}\left(F_p^{-1}(z)\right)\right] \\ &\quad \cdot \left(H_0\left(F_p^{-1}(z)\right) - H_\eta\left(F_p^{-1}(z)\right)\right)\frac{d}{dz}\left(F_p^{-1}(z)\right). \end{aligned} \tag{75}$$

Take $M = M_r > 0$ such that for all $w \in \Omega_r$ we have

$$\max\left\{|H_0(w)|, |H_\eta(w)|, \left|\frac{dH_0}{dz}(w)\right|, \left|\frac{dH_\eta}{dz}(w)\right|\right\} \leqslant M. \tag{76}$$

Note that (76) applies to $w = F_p^{-1}(z)$ for $z \in r\mathbb{D}$. We also define

$$R = R_r = \max_{w\in\partial\Omega_{(1+r)/2}} |w|.$$

Then for $\zeta \in \frac{1+r}{2}\mathbb{D}$ we have $\left|F_p^{-1}(\zeta)\right| \leqslant R$. If $z \in r\mathbb{D}$ then by the Cauchy formula we have

$$\left|\frac{d}{dz}\left(F_p^{-1}(z)\right)\right| = \left|\frac{1}{\pi(r+1)i}\oint_{\frac{1+r}{2}\partial\mathbb{D}}\frac{F_p^{-1}(\zeta)}{(\zeta-z)^2}d\zeta\right| \leqslant \max_{\zeta\in\frac{1+r}{2}\partial\mathbb{D}}\frac{\left|F_p^{-1}(\zeta)\right|}{(|\zeta|-|z|)^2} \leqslant \frac{4R}{(1-r)^2}.$$

Similarly,

$$\left|\frac{d^2}{dz^2}F_p^{-1}(z)\right| = \left|\frac{2}{\pi(r+1)i}\oint_{\frac{1+r}{2}\partial\mathbb{D}}\frac{F_p^{-1}(\zeta)}{(\zeta-z)^3}d\zeta\right| \leqslant \frac{16R}{(1-r)^3}.$$

These estimates, in conjunction with the identity (75), imply the following bound:

$$\left|\frac{d^2}{dp^2}F_p^{-1}(z)\right| \leqslant \left(2M + 2M\frac{16R}{(1-r)^3}\right)2M\frac{4R}{(1-r)^2}.$$

By the Taylor formula we deduce that

$$\left|F_p^{-1}(z) - F_0^{-1}(z) - p\,\frac{d}{dp}F_p^{-1}(z)\Big|_{p=0}\right| \leqslant C_r p^2,$$

where $C_r = \frac{8M^2R}{(1-r)^2}\left(1+\frac{16R}{(1-r)^3}\right)$. It remains to note that due to Lemma 4.3 and the identity (74), we have $\frac{d}{dp}F_p^{-1}(z)\Big|_{p=0} = (z - H_\eta(\sin z))\cos z$. □

Proof of Theorem 5.1. We will fix from now on some $r \in (9/10, 1)$. Note that since the Hermite polynomial h_5 is odd, so is H_η. Hence also F_p is odd, and therefore $a_k(p) = 0$ for even k. For $z \in r\mathbb{D}$ write $\phi(z) = (z - H_\eta(\sin z))\cos z$. Consider the power series expansions

$$\sin z = \sum_{k=0}^{\infty} b_{2k+1}z^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1},$$

and

$$\phi(z) = \sum_{k=0}^{\infty} c_{2k+1}z^{2k+1}. \tag{77}$$

By the Cauchy formula we have for every $k \in \mathbb{N} \cup \{0\}$,

$$|a_{2k+1}(p) - b_{2k+1} - pc_{2k+1}| = \left| \frac{1}{2\pi i r} \oint_{r\partial\mathbb{D}} \frac{F_p^{-1}(z) - \sin z - p\phi(z)}{z^{2k+2}} dz \right| \overset{(72)}{\leqslant} \frac{C_r p^2}{r^{2k+2}}. \tag{78}$$

Note that by Lemma 4.2 the radius of convergence of the series in (77) is at least 1, and therefore $\sum_{k=0}^{\infty} |c_{2k+1}|(9/10)^{2k+1} < \infty$. Hence,

$$\sum_{k=0}^{\infty} |a_{2k+1}(p)| \left(\frac{9}{10}\right)^{2k+1} \overset{(78)}{\geqslant} \sum_{k=0}^{\infty} \frac{(9/10)^{2k+1}}{(2k+1)!} - p \sum_{k=0}^{\infty} |c_{2k+1}| \left(\frac{9}{10}\right)^{2k+1} - \frac{C_r p^2}{r} \sum_{k=0}^{\infty} \left(\frac{9}{10r}\right)^{2k+1}$$
$$= \frac{e^{9/10} - e^{-9/10}}{2} - O(p) > 1.02 - O(p). \tag{79}$$

By continuity, it follows from (79) that provided p is small enough there exists $\gamma > 0$ satisfying the identity (63). Our goal is to prove (64), so assume for contradiction that $\gamma \leqslant \log\left(1 + \sqrt{2}\right) < 9/10$. Note that since $r \in (9/10, 1)$ we have

$$\frac{\gamma}{r} \leqslant \frac{10 \log\left(1 + \sqrt{2}\right)}{9} < \frac{49}{50}. \tag{80}$$

Fix $\varepsilon > 0$ that will be determined later. We have seen in Lemma 5.2 that $\sin\left(\frac{9}{10}\mathbb{D}\right) \subseteq \mathbb{S}$. Since H_η is analytic on $\mathbb{S}$, it follows that ϕ is analytic on $\frac{9}{10}\mathbb{D}$. Since $\gamma < 9/10$, there exists $n \in \mathbb{N}$ satisfying

$$\sum_{k=n+1}^{\infty} |c_{2k+1}| \gamma^{2k+1} < \frac{\varepsilon}{2}. \tag{81}$$

There exists $p = p(\varepsilon)$ such that for all $p \in (0, p(\varepsilon))$ we have $p|c_{2k+1}| < \frac{1}{2}|b_{2k+1}|$ for all $k \in \{0, \ldots, n\}$. In particular, we have $\mathrm{sign}(b_{2k+1} + pc_{2k+1}) = \mathrm{sign}(b_{2k+1}) = (-1)^k$. Now,

$$\left|1 - \frac{F_p^{-1}(i\gamma)}{i}\right| \overset{(63)}{=} \left| \sum_{k=0}^{\infty} \left(|a_{2k+1}(p)| - (-1)^k a_{2k+1}(p)\right) \gamma^{2k+1} \right|$$
$$\overset{(78)}{\leqslant} \sum_{k=0}^{\infty} \left| |b_{2k+1} + pc_{2k+1}| - (-1)^k \left(b_{2k+1} + pc_{2k+1}\right) \right| \gamma^{2k+1} + 2 \sum_{k=0}^{\infty} \frac{C_r p^2}{r^{2k+2}} \gamma^{2k+1}. \tag{82}$$

To estimate the two terms on the right hand side on (82), note first that

$$2 \sum_{k=0}^{\infty} \frac{C_r p^2}{r^{2k+2}} \gamma^{2k+1} \overset{(80)}{\leqslant} \frac{2C_r}{r} p^2 \sum_{k=0}^{\infty} \left(\frac{49}{50}\right)^{2k+1} \leqslant C_r' p^2, \tag{83}$$

where C_r' depends only on r. Since $p \in (0, p(\varepsilon))$ we know that for all $k \in \{0, \ldots, n\}$ we have $|b_{2k+1} + pc_{2k+1}| = (-1)^k \left(b_{2k+1} + pc_{2k+1}\right)$. Hence the first n terms of the first sum in the right hand side of (82) vanish. Therefore,

$$\sum_{k=0}^{\infty} \left| |b_{2k+1} + pc_{2k+1}| - (-1)^k \left(b_{2k+1} + pc_{2k+1}\right) \right| \gamma^{2k+1}$$
$$= \sum_{k=n+1}^{\infty} \left| |b_{2k+1} + pc_{2k+1}| - |b_{2k+1}| - (-1)^k pc_{2k+1} \right| \gamma^{2k+1} \leqslant 2p \sum_{k=n+1}^{\infty} |c_{2k+1}| \gamma^{2k+1} \overset{(81)}{<} p\varepsilon. \tag{84}$$

By substituting (83) and (84) into (82), we see that if we define $\beta = F_p^{-1}(i\gamma) - i$ then

$$|\beta| \leqslant C_r' p^2 + p\varepsilon. \tag{85}$$

Let L_0 be the Lipschitz constant of H_0 on $i + \frac{1}{2}\mathbb{D} \subseteq \mathbb{S}$ (the disc of radius $\frac{1}{2}$ centered at i). Similarly let L_η be the Lipschitz constant of H_η on $i + \frac{1}{2}\mathbb{D}$, and set $L = \max\{L_0, L_\eta\}$. It follows that $F_p = (1-p)H_0 + pH_\eta$ is L-Lipschitz on $i + \frac{1}{2}\mathbb{D}$. Due to (85), if p is small enough then $i + \beta \in i + \frac{1}{2}\mathbb{D}$, and therefore,

$$\log\left(1+\sqrt{2}\right) \geqslant \gamma = \frac{F_p(\beta+i)}{i} \geqslant \frac{F_p(i)}{i} - L|\beta| \overset{(85)}{\geqslant} \frac{(1-p)H_0(i) + pH_\eta(i)}{i} - Lp\left(C_r'p + \varepsilon\right)$$
$$= (1-p)\log\left(1+\sqrt{2}\right) + p\frac{H_\eta(i)}{i} - Lp\left(C_r'p + \varepsilon\right).$$

This simplifies to give the following estimate:

$$\frac{H_\eta(i)}{i} \leqslant \log\left(1+\sqrt{2}\right) + LC_r'p + L\varepsilon.$$

Since this is supposed to hold for all $\varepsilon > 0$ and $p \in (0, p(\varepsilon))$, we arrive at a contradiction to Theorem 4.4. □

Remark 5.2. An inspection of the proof of Theorem 1.1 shows that the only property of H_η that was used is that it is a counterexample to König's problem. In other words, assume that $f, g : \mathbb{R}^2 \to \{-1, 1\}$ are measurable functions and consider the function $H : \mathbb{S} \to \mathbb{C}$ given by

$$H(z) = \frac{1}{2\pi(1-z^2)} \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y) \exp\left(\frac{-\|x\|_2^2 - \|y\|_2^2 + 2z\langle x, y\rangle}{1-z^2}\right) dxdy.$$

Assume that $B_K(f, g) > 4\pi \log\left(1+\sqrt{2}\right)$, where B_K is König's bilinear form given in (5). Then one can repeat the proof of Theorem 1.1 with H_η replaced by H, arriving at the same conclusion.

Conjecture 5.5. *Recalling Definition 2.1 and Corollary 2.3, we conjecture that for small enough $\eta \in (0, 1)$, the pair of functions $f = g = f_\eta : \mathbb{R}^2 \to \{-1, 1\}$ is a Krivine rounding scheme for which we have $c(f_\eta, f_\eta) > \frac{2}{\pi}\log\left(1+\sqrt{2}\right)$. In other words, we conjecture that in order to prove Theorem 1.1 we do not need to use a convex combination of H_0 and H_η as we did above, but rather use only H_η itself.*

6. Proof of Theorem 1.3

For the final equality in (6), see equation (2.4) in [22]. Denote

$$M \overset{\text{def}}{=} \sup_{\substack{f,g\in L_\infty(0,\infty)\\ f,g\in[-1,1] \text{ a.e.}}} \int_0^\infty\int_0^\infty f(x)g(y)\exp\left(-\frac{x^2+y^2}{2}\right)\sin(xy)dxdy$$
$$= \sup_{\substack{f,g\in L_\infty(0,\infty)\\ f,g\in\{-1,1\} \text{ a.e.}}} \int_0^\infty\int_0^\infty f(x)g(y)\exp\left(-\frac{x^2+y^2}{2}\right)\sin(xy)dxdy. \tag{86}$$

Theorem 1.3 will follow once we show that M is attained when $f = g = 1$ or $f = g = -1$. Note that the supremum in (86) is attained at some $f, g : (0, \infty) \to \{-1, 1\}$. Indeed, let

μ be the measure on $(0,\infty)$ with density $x \mapsto e^{-x^2/2}$. If $f_n, g_n : (0,\infty) \to [-1,1]$ satisfy $\lim_{n\to\infty} \int_0^\infty \int_0^\infty f_n(x) g_n(y) \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dxdy = M$ then by passing to a subsequence we may assume that there are $f, g \in L_2(\mu)$ such that f_n converges weakly to f and g_n converges weakly to g in $L_2(\mu)$. Then $f, g \in [-1,1]$ a.e. and by weak convergence we have $\int_0^\infty \int_0^\infty f(x) g(y) \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dxdy = M$. A simple extreme point argument shows that we may also assume that $f, g \in \{-1,1\}$.

Assume from now on that $f, g : (0,\infty) \to \{-1,1\}$ are maximizers of (86), i.e.,

$$M = \int_0^\infty \int_0^\infty f(x) g(y) \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dxdy. \tag{87}$$

This implies that for almost every $x \in (0,\infty)$ we have

$$f(x) = \operatorname{sign}\left(\int_0^\infty g(y) e^{-y^2/2} \sin(xy) dy\right), \tag{88}$$

and

$$g(x) = \operatorname{sign}\left(\int_0^\infty f(y) e^{-y^2/2} \sin(xy) dy\right). \tag{89}$$

Consequently,

$$\begin{aligned} M = \int_0^\infty \left| \int_0^\infty g(y) e^{-y^2/2} \sin(xy) dy \right| e^{-x^2/2} dx \\ = \int_0^\infty \left| \int_0^\infty f(y) e^{-y^2/2} \sin(xy) dy \right| e^{-x^2/2} dx. \end{aligned} \tag{90}$$

Lemma 6.1. *Define* $I : [0,\infty) \to \mathbb{R}$ *by*

$$I(y) \stackrel{\mathrm{def}}{=} \int_0^\infty f(x) \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dx. \tag{91}$$

Then I *is* $\sqrt{2}$*-Lipschitz.*

Proof. The Lipschitz condition follows from the following simple estimate on $|I'|$:

$$\begin{aligned} |I'(y)| &\leqslant \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) |x\cos(xy) - y\sin(xy)| \, dx \\ &\leqslant \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) \sqrt{x^2+y^2} dx \\ &\leqslant \sqrt{2} e^{-y^2/2} \int_0^\infty e^{-x^2/2} \max\{x, y\} dx \\ &= \sqrt{2} e^{-y^2/2} \left(\int_0^y e^{-x^2/2} y dx + \int_y^\infty e^{-x^2/2} x dx\right) \\ &\leqslant \sqrt{2} e^{-y^2/2} \left(y^2 + e^{-y^2/2}\right) \leqslant \sqrt{2}, \end{aligned} \tag{92}$$

where in the last inequality in (92) we used the bound $e^{-t}(2t + e^{-t}) \leqslant 1$, which holds for all $t \geqslant 0$ since $e^t - e^{-t} = \sum_{k=0}^\infty \frac{2t^{2k+1}}{(2k+1)!} \geqslant 2t$. $\square$

Lemma 6.2. *For every $z \in [2/5, 4/3]$ we have*

$$\int_{z-\frac{1}{4}}^{z+\frac{1}{4}} \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dx dy > \frac{3}{20}. \tag{93}$$

Proof. Since for every $q \geqslant 0$ we have $\sin(q) \geqslant q - \frac{q^3}{6} + \frac{q^5}{120} - \frac{q^7}{5040}$,

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dx \\
&\geqslant \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)\left(xy - \frac{(xy)^3}{6} + \frac{(xy)^5}{120} - \frac{(xy)^7}{5040}\right) dx \\
&= -\frac{ye^{-y^2/2}}{105}\left(y^6 - 7y^4 + 35y^2 - 105\right) \stackrel{\mathrm{def}}{=} h(y). \qquad (94)
\end{aligned}$$

We claim that there is a unique $w \in [0,2]$ at which h' vanishes, and moreover h attains its maximum on $[0,2]$ at w. Indeed, $h'(0) = e^{-y^2/2}\left(y^8 - 14y^6 + 70y^4 - 210y^2 + 105\right)/105$, and therefore $h'(y) = 1 > 0$ and $h'(2) = -17/(7e^2) < 0$, so it suffices to show that h' can have at most one zero on $[0,2]$. To this end it suffices to show that the polynomial $p(y) = y^8 - 14y^6 + 70y^4 - 210y^2 + 105$ is monotone on $[0,2]$. This is indeed the case since for $y \in [0,2]$ we have $p'(y) = -8y^5(4-y^2) - y(52\left(y^2 - 35/13\right)^2 + 560/13) < 0$.

The above reasoning shows that if we set $G(z) = \int_{z-1/4}^{z+1/4} h(y)dy$ then G does not have local minima on $[2/5, 4/3]$. Indeed, if $z \in [2/5, 4/3]$ satisfies $G'(z) = h(z+1/4) - h(z-1/4) = 0$ then $z - 1/4 < w < z + 1/4$, implying that $h'(z-1/4) > 0$ and $h'(z+1/4) < 0$. Hence $G''(z) = h'(z+1/4) - h'(z-1/4) < 0$, so z cannot be a local minimum of G. Now, for every $z \in [2/5, 4/3]$ we have

$$\int_{z-\frac{1}{4}}^{z+\frac{1}{4}} \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dx dy \stackrel{(94)}{\geqslant} G(z) \geqslant \min\left\{G\left(\frac{2}{5}\right), G\left(\frac{4}{3}\right)\right\} > \frac{3}{20},$$

where we used the fact that the above values of G can be computed in closed form, for example $G(4/3) = (19047383/313528320)e^{-169/288} + (131938921/313528320)e^{-361/288} > 0.153$. $\square$

We will consider the following two quantities:

$$M_0 \stackrel{\mathrm{def}}{=} \int_0^\infty \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) \sin(xy) dx dy = \frac{\log\left(1+\sqrt{2}\right)}{\sqrt{2}} = 0.6232..., \tag{95}$$

and

$$M_1 \stackrel{\mathrm{def}}{=} \int_0^\infty \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) |\sin(xy)| dx dy. \tag{96}$$

Our goal is to show that $M = M_0$. Clearly $M_0 \leqslant M_1$. Our next (technical) lemma shows that M_1 is actually quite close to M_0.

Lemma 6.3. $M_1 - M_0 < \frac{1}{20}$.

Proof. Since the integral in (96) converges very quickly and can therefore be computed numerically with high precision, Lemma 6.3 does not have much content. Nevertheless, we wish to explain how to reduce this lemma to an evaluation of an integral that can be

computed in closed form. Let p_n be the Taylor polynomial of degree $2n-1$ of the function $x \mapsto \sqrt{1-x}$, i.e.,

$$p_n(x) = \sum_{k=0}^{2n-1} (-1)^k \binom{1/2}{k} x^k = 1 + \sum_{k=1}^{2n-1} \frac{\prod_{j=0}^{k-1}(2j-1)}{2^k k!} x^k.$$

Then $\sqrt{1-x} \leqslant p_n(x)$ for all $x \in (-1,1)$ and $n \in \mathbb{N}$.

Now,

$$\begin{aligned} M_1 &= \int_0^\infty \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)\sqrt{\frac{1-\cos(2xy)}{2}}dxdy \\ &\leqslant \frac{1}{\sqrt{2}}\int_0^\infty \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right) p_n(\cos(2xy))dxdy. \quad (97) \end{aligned}$$

For $j \in \mathbb{N}$ the integral $\int_0^\infty \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)(\cos(2xy))^j dxdy$ can be computed in closed form (it equals π times a linear combination with rational coefficients of square roots of integers). One can therefore explicitly evaluate the integral in right hand side of (97) for $n = 11$, obtaining the bound $M_1 < 0.671 < M_0 + 0.05$. $\square$

Lemma 6.4. *f and g are constant on the interval* $[2/5, 4/3]$.

Proof. Assume for contradiction that g is not constant on $[2/5, 4/3]$. By (89) we know that $g = \operatorname{sign}(I)$, where I is given in (91). Hence there is some $z \in [2/5, 4/3]$ such that $I(z) = 0$. By Lemma 6.1 we therefore know that $|I(y)| \leqslant \sqrt{2}|y-z|$ for all $y \in [0,\infty)$. Hence,

$$\begin{aligned} M_0 \leqslant M &\overset{(90)}{\leqslant} \int_{[0,z-1/4]\cup[z+1/4,\infty)} \int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)|\sin(xy)|dxdy + \int_{z-\frac14}^{z+\frac14} |I(y)|dy \\ &\leqslant M_1 - \int_{z-\frac14}^{z+\frac14}\int_0^\infty \exp\left(-\frac{x^2+y^2}{2}\right)|\sin(xy)|dxdy + \frac{\sqrt{2}}{16} \overset{(93)}{<} M_1 - \frac{3}{20} + \frac{\sqrt{2}}{16} < M_1 - \frac{3}{50}. \end{aligned}$$

This contradicts Lemma 6.3. $\square$

Before proceeding to the proof of Theorem 1.3, we record one more elementary lemma.

Lemma 6.5. *For every $a > 0$ we have*

$$\int_a^\infty e^{-x^2/2}dx < \frac{16}{3e^2a^3}. \quad (98)$$

Proof. Set $c = 16/(3e^2)$ and $\psi(a) = \frac{c}{a^3} - \int_a^\infty e^{-x^2/2}dx$. Since $\lim_{a\to\infty}\psi(a) = 0$, it suffices to show that ψ is decreasing on $(0,\infty)$. Because $\psi'(a) = e^{-a^2/2} - \frac{3c}{a^4}$, our goal is to show that $e^{a^2/2} \geqslant a^4/(3c)$, or equivalently that $a^2 \geqslant 8\log a - 2\log(3c)$. Set $\rho(a) = a^2 - 8\log a + 2\log(3c)$. Since $\rho'(a) = 2a - 8/a$, the minimum of ρ is attained at $a = 2$. We are done, since by the choice of c we have $\rho(2) = 0$. $\square$

Proof of Theorem 4.4. Due to Lemma 6.4, we may assume from now on that $f(x) = 1$ for all $x \in [2/5, 4/3]$. Our goal is to show that under this assumption $f = g = 1$. We shall achieve this in a few steps.

We will first show that $f(y) = g(y) = 1$ for all $y \in [0, 4/3]$. To see this, fix $y \in (0, 1/2]$ and note that the function $x \mapsto \sin(xy)$ is concave on $[0, 4/3]$ (since in this range we have $xy \leqslant 2/3 \leqslant \pi/2$). It follows that for $x \in [0, 4/3]$ we have $\sin(xy) \geqslant \frac{\sin(4y/3)}{4/3}x$ and for $x \in [4/3, \infty)$ we have $|\sin(xy)| \leqslant \frac{\sin(4y/3)}{4/3}x$. Using this fact, the fact that $|\sin(xy)| \leqslant xy$ for all $x \geqslant 0$, and the fact that $f = 1$ on $[2/5, 4/3]$, we deduce that

$$\begin{aligned}
&\int_0^\infty e^{-x^2/2}\sin(xy)f(x)dx \\
&\qquad \geqslant -\int_0^{\frac{2}{5}} e^{-x^2/2}xydx + \int_{\frac{2}{5}}^{\frac{4}{3}} e^{-x^2/2}\frac{\sin(4y/3)}{4/3}xdx - \int_{\frac{4}{3}}^\infty e^{-x^2/2}\frac{\sin(4y/3)}{4/3}xdx \\
&\qquad = -\left(1-e^{-2/25}\right)y + \frac{\sin(4y/3)}{4/3}\left(e^{-2/25} - 2e^{-8/9}\right) \stackrel{\mathrm{def}}{=} r(y). \qquad (99)
\end{aligned}$$

Note that for $z \in [0, 1/2]$ we have

$$\begin{aligned}
r'(z) &= \cos\left(\frac{4z}{3}\right)\left(e^{-2/25} - 2e^{-8/9}\right) - \left(1-e^{-2/25}\right) \\
&\geqslant \cos\left(\frac{2}{3}\right)\left(e^{-2/25} - 2e^{-8/9}\right) - \left(1-e^{-2/25}\right) > 0.002 > 0.
\end{aligned}$$

Hence r in increasing on $[0, 1/2]$, and in particular $r(y) > r(0) = 0$. By (89) and (99), it follows that $g(y) = 1$ for all $y \in [0, 1/2]$. Since Lemma 6.4 tells us that g is constant on $[2/5, 4/3]$, it follows that $g = 1$ on $[0, 4/3]$. We my now repeat the above argument with the roles of f and g interchanged, deducing that $f = g = 1$ on $[0, 4/3]$.

Our next goal is to show that $f = g = 1$ on $[0, 3\pi/4]$. We already know that $f = g = 1$ on $[0, 4/3]$, so assume that $y \in [4/3, 3\pi/4]$. Then $4y/3 \geqslant (4/3)^2 > \pi/2$ and $4y/3 \leqslant \pi$, implying that

$$\begin{aligned}
\int_0^{\frac{4}{3}} f(x)e^{-x^2/2}\sin(xy)dx &\geqslant \int_0^{\frac{\pi}{2y}} e^{-x^2/2}\sin(xy)dx \stackrel{(*)}{\geqslant} \int_0^{\frac{\pi}{2y}} e^{-x^2/2}\frac{2xy}{\pi}dx \\
&= \frac{2y}{\pi}\left(1-e^{-\pi^2/(8y^2)}\right) \stackrel{(**)}{\geqslant} \frac{2y}{\pi}\left(\frac{\pi^2}{8y^2} - \frac{\pi^4}{128y^4}\right) = \frac{\pi}{4y} - \frac{\pi^3}{64y^3} \stackrel{(***)}{\geqslant} \frac{8}{27}, \qquad (100)
\end{aligned}$$

where in $(*)$ we used the fact that $\sin(z) \geqslant 2z/\pi$ for $z \in [0, \pi/2]$, in $(**)$ we used the elementary inequality $1 - e^{-z} \geqslant z - z^2/2$, which holds for all $z \geqslant 0$, and in $(***)$ we used the fact that the function $y \mapsto \frac{\pi}{4y} - \frac{\pi^3}{64y^3}$ has a unique local maximum on $(1, \infty)$, which implies that its minimum on $[4/3, 3\pi/4]$ is attained at the endpoints, and therefore its minimum on $[4/3, 3\pi/4]$ equals $8/27$.

Now,

$$\begin{aligned}
\int_0^\infty f(x)e^{-x^2/2}\sin(xy)dx &\stackrel{(100)}{\geqslant} \frac{8}{27} - \int_{\frac{4}{3}}^\infty e^{-x^2/2}dx = \frac{8}{27} - \left(\sqrt{\frac{\pi}{2}} - \int_0^{\frac{4}{3}} e^{-x^2/2}dx\right) \\
&\geqslant \frac{8}{27} - \sqrt{\frac{\pi}{2}} + \int_0^{\frac{4}{3}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)dx = \frac{302572}{229635} - \sqrt{\frac{\pi}{2}} > 0. \qquad (101)
\end{aligned}$$

Using (89) we deduce from (101) that $g(y) = 1$. By symmetry the same argument applies to f, implying that $f = g = 1$ on $[0, 3\pi/4]$.

Let $A \geqslant 3\pi/4$ be the supremum over those $a > 0$ such that $f = g = 1$ on $[0, a]$. Our goal is to show that $A = \infty$, so assume for contradiction that A is finite.

Note that for every $y > 0$ and every $k \in \mathbb{N} \cup \{0\}$,

$$\int_{\frac{2k\pi}{y}}^{\frac{2(k+1)\pi}{y}} e^{-x^2/2} \sin(xy) dx = \int_{\frac{2k\pi}{y}}^{\frac{(2k+1)\pi}{y}} \left(e^{-x^2/2} - e^{-(x+\pi/y)^2/2}\right) \sin(xy) dx \geqslant 0. \tag{102}$$

It follows that if $A \in [2k\pi/y, 2(k+1)\pi/y]$ then

$$\int_{\frac{2k\pi}{y}}^{A} e^{-x^2/2} \sin(xy) dx \geqslant 0. \tag{103}$$

To check (103), note that for $A \in [2k\pi/y, (2k+1)\pi/y]$ the integrand in (103) is nonnegative, and for $A \in [(2k+1)\pi/y, 2(k+1)\pi/y]$ the integral in (103) is at least the integral in the left hand side of (102). Assume from now on that $y > A$ and note that since $A \geqslant 3\pi/4$ we have $3\pi/(2y) < A$. This implies the following bound:

$$\begin{aligned}
\int_0^{\min\left\{\frac{2\pi}{y}, A\right\}} e^{-x^2/2} \sin(xy) dx &\geqslant \int_0^{\frac{2\pi}{y}} e^{-x^2/2} \sin(xy) dx \\
&= \int_0^{\frac{\pi}{y}} \left(e^{-x^2/2} - e^{-(x+\pi/y)^2/2}\right) \sin(xy) dx \\
&= \int_0^{\frac{\pi}{y}} e^{-x^2/2} \left(1 - e^{-\pi x/y} e^{-(\pi/y)^2/2}\right) \sin(xy) dx \\
&\geqslant \left(1 - e^{-(\pi/y)^2/2}\right) \int_0^{\frac{\pi}{2y}} e^{-x^2/2} \frac{2xy}{\pi} dx \\
&= \frac{2y}{\pi} \left(1 - e^{-\pi^2/(2y^2)}\right) \left(1 - e^{-\pi^2/(8y^2)}\right) \\
&\geqslant \frac{2y}{\pi} \left(\frac{\pi^2}{2y^2} - \frac{\pi^4}{8y^4}\right) \left(\frac{\pi^2}{8y^2} - \frac{\pi^4}{128y^4}\right) \geqslant \frac{5\pi^3}{81y^3},
\end{aligned} \tag{104}$$

where in the last inequality of (104) we used the fact that $y > A \geqslant 3\pi/4$.

A combination of (102), (103) and (104) with the fact that $f = 1$ on $[0, A)$ shows that

$$\int_0^A f(x) e^{-x^2/2} \sin(xy) dx \geqslant \frac{5\pi^3}{81y^3},$$

and therefore, by Lemma 6.5,

$$\int_0^\infty f(x) e^{-x^2/2} \sin(xy) dx \geqslant \frac{5\pi^3}{81y^3} - \int_A^\infty e^{-x^2/2} dx \overset{(98)}{\geqslant} \frac{5\pi^3}{81y^3} - \frac{16}{3e^2 A^3}. \tag{105}$$

The right hand side of (105) is positive provided $y \leqslant 5A/4$. By (89) this means that $g = 1$ on $[A, 5A/4]$, and hence also on $[0, 5A/4]$. By symmetry, $f = 1$ on $[0, 5A/4]$ as well, contradicting the definition of A. $\square$

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University of Toronto, 10 King's College Road, Toronto, ON M5S 3G4.
E-mail address: `mbraverm@cs.toronto.edu`

IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598.
E-mail address: `konstantin@us.ibm.com`

Toyota Technological Institute at Chicago, 6045 S. Kenwood Ave., Chicago, IL 60637.
E-mail address: `yury@ttic.edu`

Courant Institute, 251 Mercer Street, New York, NY 10012.
E-mail address: `naor@cims.nyu.edu`